Examination: Computational Geophysics
June 27, 2014, 13:15-16:15

Problem 1

A model for Rayleigh-Benard (R-B) convection of an *isoviscous* fluid layer in the *Boussinesq-approximation* can be written in terms of a *stream-function-vorticity* formulation by the following coupled equations,

$$\nabla^2\omega = -Ra\partial_x T \tag{1}$$

$$\nabla^2\psi = \omega \tag{2}$$

$$\frac{\partial T}{\partial t} + \mathbf{u}\cdot\nabla T = \nabla^2 T \tag{3}$$

We consider a 2-D rectangular domain $V = [0, L] \times [0, 1]$ with free-slip impermeable boundary conditions formulated as,

$$\omega(\mathbf{x}) = 0, \psi(\mathbf{x}) = 0, \mathbf{x} \in \partial V \tag{4}$$

The model equations are discretized using a *finite element* method on a grid with N_e *finite elements* spanned by N_p nodal points $\mathbf{x}_J$, $J = 1, \ldots, N_p$.

1. Boundary conditions for the temperature field T in the R-B convection problem differ from the conditions (4) applied to the ω, ψ problem. Specify apropriate boundary conditions for the temperature field T assuming symmetrical continuation of the finite domain in the horizontal direction.

2. Discuss the different number of degrees of freedom of the discretized solution fields $\Omega_J = \omega(\mathbf{x}_J)$, $\Psi_J = \psi(\mathbf{x}_J)$ on the one hand and $T_J = T(\mathbf{x}_J)$ on the other, as a result of the different boundary conditions for these fields.

3. In the following *finite element* equations are considered for the coupled system (1), (2), (3). This item 3. concerns equation (1). Equations (2) and (3) are dealt with in items 4., 5., 6.

   - First derive explicit expressions for the different $\mathbf{A}, \mathbf{F}$ terms in the *finite element* equation (5), corresponding to the vorticity equation (1), using a *Galerkin* method.

     $$\mathbf{A}\boldsymbol{\Omega} = \mathbf{F} \tag{5}$$

     Note: Give a derivation of the algebraic equations, starting from the Galerkin principle applied to (1), using general finite elements and corresponding basis functions based on piece wise Lagrange interpolation per element.

   - Show that the right-hand side vector $\mathbf{F}$ can be written as $F_I = \sum_J R_{IJ} T_J$, $I = 1, 2, \ldots$, and express the matrix R_{IJ} interms of the *finite element* basis functions.

- Discuss the structure of the matrix of the resulting system of equations. (Symmetry, sparsity structure).

4. Derive a similar system of *finite element* equations for the stream function equation (2), with the same matrix as in (5).

$$\mathbf{A\Psi} = \mathbf{G} \tag{6}$$

Explain why the matrices in (5) and (6) are identical and how the right hand side vector $\mathbf{G}$ is defined in terms of the vorticity field ω and *finite element* basis functions.

5. For the energy equation (3) we consider the special case of a steady state model, where the time derivative is dropped in (3).

   - Derive the corresponding *finite element* equation,

   $$\mathbf{ST} = \mathbf{R} \tag{7}$$

   for the energy transport equation, using a Galerkin method applied to (3). Show in your derivation how the advection term $\mathbf{u} \cdot \nabla T$ in (3) is included in the application of the Galerkin method, to derive an expression for the stiffness matrix $\mathbf{S}$.

   - Also derive an expression for the right-hand side vector $\mathbf{R}$ for this model.

6. The velocity field $\mathbf{u}$ in (3) is expressed in the stream function as,

$$\mathbf{u} = (\partial_z \psi, -\partial_x \psi) = -(\partial_y \psi, \partial_x \psi) \tag{8}$$

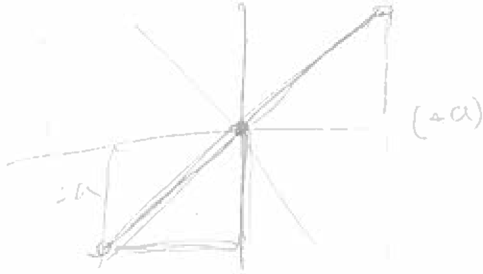
Assume that 2-D linear triangular elements are used for both the stream function and the vorticity fields.

Derive the following expression for the contribution to the stiffness matrix in (7) from the advective energy transport,

$$S_{IJ}^{adv} = \sum_{K=1}^{N_e} \int_{e_K} N_I \left(U_{1K} \frac{\partial N_J}{\partial x} + U_{2K} \frac{\partial N_J}{\partial y} \right) dV \tag{9}$$

where U_{1K} and U_{2K} are the components of the flow velocity field, that are piece wise constant per element *(why?)*.

7. The discretized equations derived above can now be used to solve the steady state R-B convection problem. Explain why these three systems of equations are coupled in a non-linear way and discribe an iterative procedure for the solution of the coupled equations.



Problem 2

Steady state elastic deformation problems can be formulated with the elastostatic equation,

$$\partial_j \sigma_{ij} + \rho F_i = 0 \tag{10}$$

We consider a 2-D plain strain configuration where the displacement is in the vertical plane, $\mathbf{u} = (u, v, 0)$. For a linear elastic medium the constitution equation relating the stress and strain tensors can be written in the following matrix vector scheme,

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{pmatrix} = \begin{pmatrix} \lambda + 2\mu & \lambda & 0 \\ \lambda & \lambda + 2\mu & 0 \\ 0 & 0 & \mu \end{pmatrix} \begin{pmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{12} \end{pmatrix} \tag{11}$$

where the strain tensor is defined in terms of the displacement field as, $\epsilon_{ij} = 1/2\,(\partial_i u_j + \partial_j u_i)$. We apply the above to model the elastic deformation of a column under its own weight. We assume that the displacement on the vertical boundaries is constrained to the vertical direction with a zero tangential stress condition (free slip). For the horizontal bottom and top boundaries of the 2-D rectangular domain we define respectively, zero displacement (rigid) and zero traction (free boundary). This can be formulated as a 1-D scalar problem with $\mathbf{u} = (0, v, 0)$.

1. Show that the elastostatic equation for this problem reduces to the following ordinary differential equation for the vertical displacement v,

   $$\frac{d}{dy}\left((\lambda + 2\mu)\frac{dv}{dy}\right) = -\rho g \tag{12}$$

   where the gravity acceleration has been substituted for the body force field.

2. For the case with uniform elasticity coefficients λ, μ a polynomial solution of degree two is derived in the lecture notes. For cases with variable elasticity coefficients numerical models for the above problem can be obtained with a finite element method.

   Derive expressions for the stiffness matrix $\mathbf{S}$ and right-hand side vector $\mathbf{F}$ of the corresponding finite element equation $\mathbf{SV} = \mathbf{F}$, where $V_J = v(\mathbf{x}_J)$ is the finite element solution vector. Assume general finite elements and corresponding basis functions N_J here.

3. Discuss the application of linear (2-node) elements for the equation of the last item.

$\partial_x \sigma_{ix} + \partial_y \sigma_{iy} = -\rho F_i$

$\partial_y \sigma_{iy} = -\rho F_i$