DIVA: End-term test 2015

Write your name and student number, and write READABLE.

(1) 1. Solve the separable 1st order differential equation

$(x + xy)y' + y = 0$ with the boundary condition $y(1) = 1$

(1) 2. We have the inhomogeneous 1st order differential equation

$dx + (x - e^{y})dy = 0$

First, solve the homogeneous equation. *Hint: solve for x in terms of y.*

Then require $C \to C(y)$ and solve the inhomogeneous equation

3. The solution of an inhomogenous 2nd order LDE with constant coefficients $y'' + py' + qy = r(x)$ (p,q constant) is $y(x) = y_c(x) + y_p(x)$

With $y_c(x) = c_1 y_1(x) + c_2 y_2(x)$ and $y_p(x) = u(x)y_1(x) + v(x)y_2(x)$

Under certain conditions it appears that:

$$u(x) = \int \frac{-y_2 r(x)}{W(x)} \text{ en } v(x) = \int \frac{y_1 r(x)}{W(x)} \text{ met } W(x) = y_1 y_2' - y_1' y_2$$

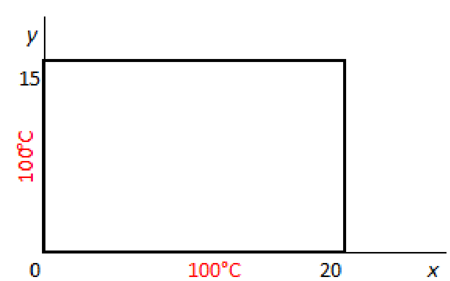
We can now solve any 2nd order LDE *(5 stappen plan)*

(1) a. $y'' + 16y = 16\sin 4x$ **(1)** b. $2y'' + y' = 2x$

(2) 4. *Laplace:* $\nabla^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$ with solutions $T = XY = \begin{Bmatrix} e^{ky} \\ e^{-ky} \end{Bmatrix} \begin{Bmatrix} \sin kx \\ \cos kx \end{Bmatrix}$

The steady state temperature distribution in a metal plate 10 cm square if one side (along x-axis) is held at 100°C and the other three sides at 0°C has a solution $T = \sum_{odd\ n}^{\infty} \frac{400}{n\pi \sinh n\pi} \sinh \frac{n\pi}{10}(10 - y) \sin \frac{n\pi}{10} x$

Now consider a metal plate 15 x 20 cm square:



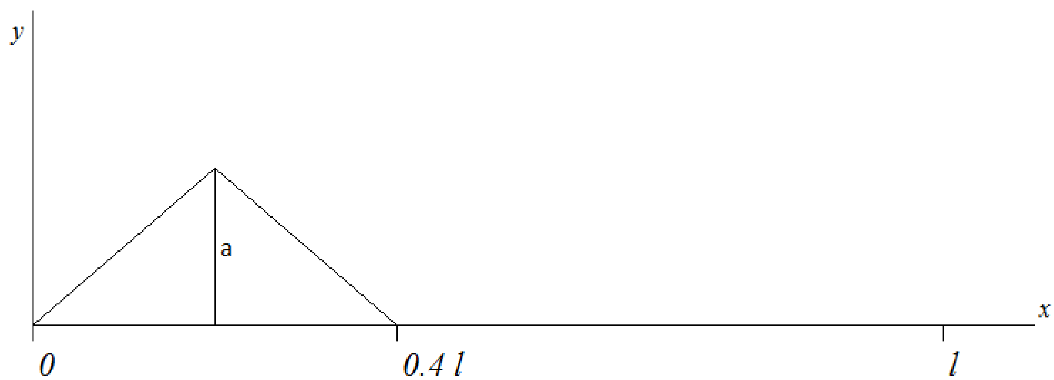
Find the steady state temperature distribution in this plate

(2) 5. *Diffusie equation:* $\nabla^2 u = \frac{1}{\alpha^2}\frac{\partial u}{\partial t}$ with solutions $u = FT = \begin{Bmatrix} e^{-k^2\alpha^2 t}\sin kx \\ e^{-k^2\alpha^2 t}\cos kx \end{Bmatrix}$

A bar of length l is initially at 0°C. From $t=0$ on, the $x=0$ end is held at T_1°C and the $x=l$ at T_2°C. Find the time-dependent temperature distribution.

(2) 6. *Wave equation:* $\nabla^2 y = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$ with solutions $y = XT = \begin{Bmatrix} \sin kx \\ \cos kx \end{Bmatrix}\begin{Bmatrix} \sin \omega t \\ \cos \omega t \end{Bmatrix}$

A string of length *l* has a zero initial velocity and a displacement as shown (plucked string problem)



Find the displacement as a function of *x* and *t*

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1 $\boxed{8.2.12}$. $(x+xy)y'+y=0 \rightarrow \frac{dy}{dx} = \frac{-y}{x(1+y)}$

$$\int \frac{1+y}{y}dy = \int -\frac{1}{x}dx \rightarrow \int \frac{1}{y}dy + \int dy = \int -\frac{1}{x}dx$$

$$\ln|y| + y = -\ln|x| + c \rightarrow \ln|y| + \ln|x| = -y + c$$

$$\ln|xy| = -y + c \rightarrow xy = e^{-y+c} = ke^{-y}$$

$$xye^{y} = k \qquad y=1 \text{ if } x=1 \rightarrow k = e$$

$$\rightarrow x = \frac{e}{ye^{y}}$$

2 $dx + (x - e^{y})dy = 0$

$$\frac{dx}{dy} + x = e^{y}$$

1) $\frac{dx}{dy} + x = 0 \rightarrow \int \frac{1}{x}dx = -\int dy$

$$\ln|x| = -y + c \rightarrow x_c = e^{-y+c_1} = ce^{-y}$$

2). Suppose $C = C(y) \rightarrow x = C(y)e^{-y}$

$$\frac{dx}{dy} + x = e^{y} \rightarrow c'(y)e^{-y} \cancel{- c(y)e^{-y}} \cancel{+ c(y)e^{-y}} = e^{y}$$

$$\rightarrow c'(y) = e^{2y} \rightarrow c(y) = \frac{1}{2}e^{2y} \rightarrow x_p = \frac{1}{2}e^{2y} \cdot e^{-y} = \frac{1}{2}e^{y}$$

$$\boxed{x = x_c + x_p = ce^{-y} + \frac{1}{2}e^{y}}$$

3 a) $y'' + 16y = 16 \sin 4x$

see solution to $\boxed{8.6.17}$ homework

and replace $\mathrm{Re}[\cdots]$ by $\mathrm{Im}[\cdots]$

then: $y_p(x) = \mathrm{Im}\left[-2ix\cos 4x + 2x \sin 4x + \frac{1}{4}\cos 4x + \frac{1}{4} i \sin 4x\right]$

$$= -2x\cos 4x + \frac{1}{4}\sin 4x.$$

hence: $y(x) = y_c + y_p$

$$= (d_1 + \tfrac{1}{4})\sin 4x + d_2 \cos 4x + 2x\cos 4x$$

$$= C_1 \sin 4x + C_2 \cos 4x + 2x \cos 4x$$

b) $\boxed{8.6.22}$ $\quad 2y'' + y' = 2x \quad$ First homogeneous:

$2m^2 + m = 0 \rightarrow 2m(m + \frac{1}{2}) = 0 \rightarrow m_1 = 0, \; m_2 = -\frac{1}{2}$

$\boxed{y_c} = c_1 e^0 + c_2 e^{-x/2} = \boxed{C_1 + C_2 e^{-x/2}} \rightarrow y_1 = 1, \; y_2 = e^{-x/2}$

inhomogenous: $y_p = u(x) + v(x)e^{-x/2}$

$W(x) = 1 \cdot -\frac{1}{2}e^{-x/2} - 0 = -\frac{1}{2}e^{-x/2}$

$u(x) = \int \frac{e^{-x/2} \cdot 2x}{-\frac{1}{2}e^{-x/2}}\,dx = \int 4x\,dx = 2x^2$

$v(x) = \int \frac{1 \cdot 2x}{-\frac{1}{2}e^{-x/2}}\,dx = -4\int xe^{x/2} = -4\left\{2xe^{x/2} - \int 2e^{x/2}\,dx\right\}$

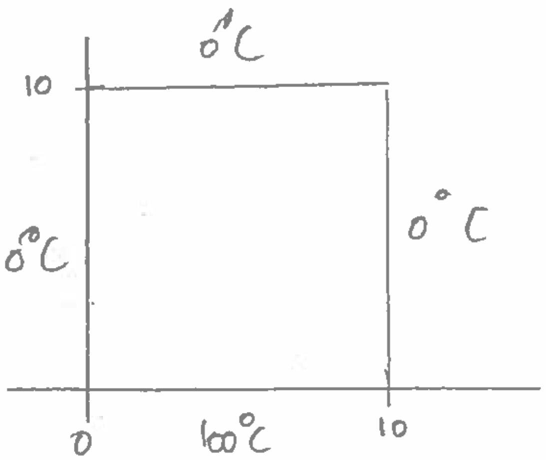
$= -8xe^{x/2} + 8 \cdot 2e^{x/2} = -8xe^{x/2} + 16e^{x/2}$

$y_p = 2x^2 + (-8xe^{x/2} + 16e^{x/2})e^{-x/2} = 2x^2 - 8x + 16$

$y = y_c + y_p = C_1 + C_2 e^{-x/2} + 2x^2 - 8x + 16$

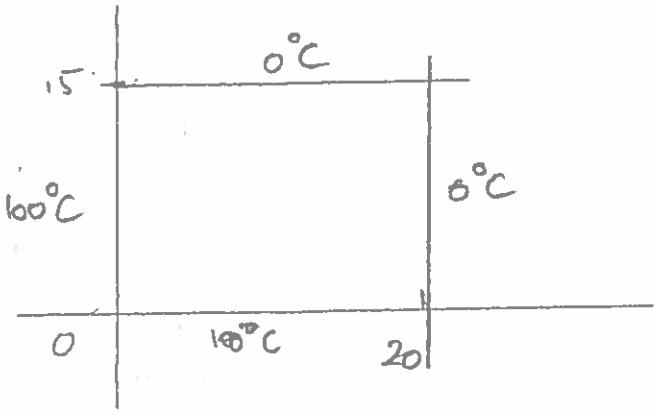
$\rightarrow y = A + Be^{-x/2} + x^2 - 4x.$ $\quad \left\{ \begin{array}{l} A = \frac{C_1}{2} + 16 \\ B = \frac{C_2}{2} \end{array} \right.$

(4)



Assignment $\boxed{3.2.10}$ gave the answer given in the exam. See also $\boxed{3.2.11}$ which was a home work assignment.

But now we have: (a variation of $\boxed{3.2.12}$)



So the solution for the x-axis 100°C becomes

$$T_1(x,y) = \sum_{\text{odd}\,n} \frac{400}{n\pi \sinh\left(\frac{3}{4}n\pi\right)} \cdot \sinh \frac{n\pi}{20}(15-y) \cdot \sin \frac{n\pi}{20}x$$

since $y=15$ must give $T=0°C$.

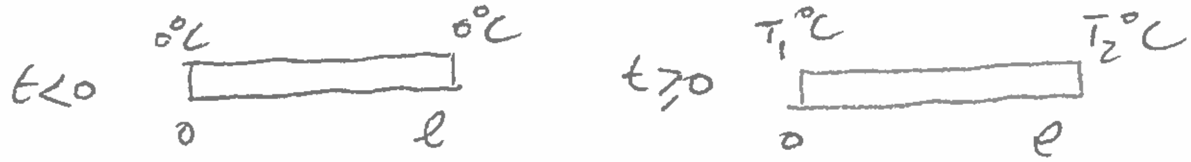
Similarly the solution for the y-axis 100°C is then:

$$T_2(x,y) = \sum_{\text{odd}\,n} \frac{400}{n\pi \sinh\left(\frac{4}{3}n\pi\right)} \cdot \sinh \frac{n\pi}{15}(20-x) \sin \frac{n\pi}{15}y$$

Hence, the final solution is $T(x,y) = T_1(x,y) + T_2(x,y)$

Note: in $T_1(x,y)$ the $\sinh\left(\frac{3}{4}n\pi\right)$ derives from the ratio $\frac{15}{20}$, in the $T_2(x,y)$ the $\sinh\left(\frac{4}{3}n\pi\right)$ derives from the ratio $\frac{20}{15}$

5 $\nabla^2 u = \frac{1}{\alpha^2}\frac{\partial u}{\partial t}$ met $u = FT = \begin{cases} e^{-k^2\alpha^2 t} \sin kx \\ e^{-k^2\alpha^2 t} \cos kx \end{cases}$



This is similar to Example 1 in Boas, pp. 629

$$\left.\begin{aligned} ax+b &= T_1 \text{ op } x=0 \rightarrow b = T_1 \\ ax+b &= T_2 \text{ op } x=l \rightarrow al+b = T_2 \end{aligned}\right\} a = (T_2-T_1)/l$$

Hence; $u_0 = 0$, $u_f = \frac{T_2-T_1}{l}x + T_1$

$$\left.\begin{aligned} &1.\ \cos \text{ terms disappear} \\ &2.\ \sin kl = 0 \rightarrow k = \frac{n\pi}{l} \end{aligned}\right\} u = e^{-\left(\frac{n\pi}{l}\right)^2\alpha^2 t} \cdot \sin\frac{n\pi}{l}x$$

3. At $t=0$ is $u = u_0 \rightarrow \sum \ldots = u_0 - u_f = -u_f$.

Therefore: $b_n = \frac{2}{l}\int_0^l f(x) \sin kx\, dx$

$$\rightarrow b_n = -\frac{2}{l}\int_0^l \left(\frac{T_2-T_1}{l}x + T_1\right)\sin\frac{n\pi}{l}x\, dx$$

$$= -\frac{2(T_2-T_1)}{l^2}\int_0^l x \sin\frac{n\pi}{l}x\, dx - \frac{2T_1}{l}\int_0^l \sin\frac{n\pi}{l}x\, dx$$

Partial integration first part:

$$\rightarrow b_n = -\frac{2(T_2-T_1)}{l^2}\left[\frac{-x\, l}{n\pi}\cos\frac{n\pi}{l}x\,\Big|_0^l + \int_0^l \frac{l}{n\pi}\cos\frac{n\pi}{l}x\, dx\right.$$

$$= -\frac{2(T_2-T_1)}{l^2}\left[-\frac{xl}{n\pi}\cos\frac{n\pi}{l}x + \cancel{\frac{l^2}{(n\pi)^2}\sin\frac{n\pi}{l}x}\right]_0^l$$

(zero)

$$= \frac{-2(T_2-T_1)}{l^2}\left[-\frac{l^2}{n\pi}\cos n\pi\right] = \frac{2(T_2-T_1)}{n\pi}(-1)^n$$

5

ctd

Second part:

$$-\frac{2T_1}{\ell}\int_0^{\ell}\sin\frac{n\pi}{\ell}x\,dx = -\frac{2T_1}{\ell}\cdot\frac{-\ell}{n\pi}\left[\cos\frac{n\pi}{\ell}x\right]_0^{\ell}$$

$$= \frac{2T_1}{n\pi}\left[\cos n\pi - 1\right] = \frac{2T_1}{n\pi}\left[(-1)^n - 1\right]$$

$$= \begin{cases} 0 & n \text{ even} \\ \frac{-4T_1}{n\pi} & n \text{ odd} \end{cases}$$

Hence:

simplified:

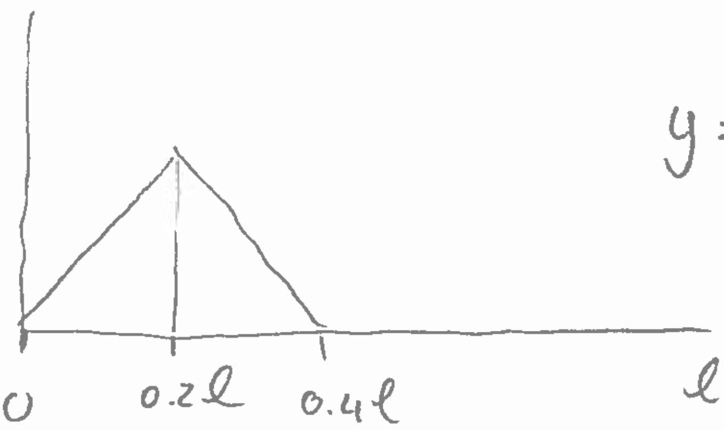
$$b_n = \frac{2(T_2-T_1)}{n\pi}(-1)^n - \left.\frac{4T_1}{n\pi}\right|_{n\,\text{odd}} = \frac{2}{n\pi}\left(T_2(-1)^n - T_1\right)$$

$$\rightarrow \quad u = u_f + \frac{2(T_2-T_1)}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}e^{-k^2\alpha^2 t}\cdot\sin\frac{n\pi}{\ell}x$$

$$- \frac{4T_1}{\pi}\sum_{n\,\text{odd}}^{\infty}\frac{1}{n}e^{-k^2\alpha^2 t}\sin\frac{n\pi}{\ell}x$$

$$\rightarrow \boxed{u = T_1 + \frac{T_2-T_1}{\ell}x + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{T_2(-1)^n - T_1}{n}e^{-k^2\alpha^2 t}\cdot\sin\frac{n\pi}{\ell}x}$$

6



$$y = XT = \begin{Bmatrix} \sin kx \\ \cos kx \end{Bmatrix} \begin{Bmatrix} \sin \omega t \\ \cos \omega t \end{Bmatrix}$$

This is identical to $\boxed{13.4.2}$ with
0.2ℓ / 0.4ℓ instead of ¼ℓ / ½ℓ, and $h = a$.

$$f(x) = \begin{cases} \frac{5a}{\ell} x & 0 < x < \frac{1}{5}\ell \\ 2a - \frac{5a}{\ell} x & \frac{1}{5}\ell < x < \frac{2}{5}\ell \\ 0 & \frac{2}{5}\ell < x < \ell \end{cases}$$

if $t = 0$ then $\frac{\partial y}{\partial t} = 0$ so $\sin \omega t$ terms disappear

at $x = 0 \rightarrow y = 0$ so $\cos kx$ terms must go.

at $x = \ell \rightarrow y = 0 \rightarrow \sin kx = 0 \rightarrow k = \frac{n\pi}{\ell}$

$$y(x,t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{\ell} x \cdot \cos \frac{n\pi}{\ell} vt = f(x)$$

at $t = 0$ $\cos \frac{n\pi}{\ell} vt = 1$ and hence:

$$b_n = \frac{2}{\ell} \int_0^{\ell} y(x,0) \sin \frac{n\pi}{\ell} x \, dx =$$

$$= \frac{2}{\ell} \int_0^{\ell/5} \frac{5a}{\ell} x \sin \frac{n\pi}{\ell} x \, dx + \frac{2}{\ell} \int_{\ell/5}^{2\ell/5} \left(2a - \frac{5a}{\ell} x\right) \sin \frac{n\pi}{\ell} x \, dx$$

6 ctd

$$b_n = \frac{2}{\ell}\cdot\frac{5a}{\ell}\left[\frac{-\ell}{n\pi}x\cos\frac{n\pi}{\ell}x + \frac{\ell^2}{(n\pi)^2}\sin\frac{n\pi}{\ell}x\right]_0^{\ell/5}$$

$$+\frac{2}{\ell}\left[\left(2a-\frac{5a}{\ell}x\right)\frac{-\ell}{n\pi}\cos\frac{n\pi}{\ell}x - \left(\frac{5a}{\ell}\right)\frac{\ell^2}{(n\pi)^2}\sin\frac{n\pi}{\ell}x\right]_{\ell/5}^{2\ell/5}$$

$$= \frac{10a}{\ell^2}\left[\frac{-\ell^2}{n\pi 5}\cos\frac{n\pi}{5} + \frac{\ell^2}{(n\pi)^2}\sin\frac{n\pi}{5}\right]$$

$$+\frac{2}{\ell}\left[\cancel{\underbrace{\left(2a-\frac{5a}{\ell}\frac{2\ell}{5}\right)}_{2a-2a=0}\frac{-\ell}{n\pi}\cos\frac{2n\pi}{5}} - 0 - \frac{5a}{\ell}\frac{\ell^2}{(n\pi)^2}\sin\frac{2n\pi}{5}\right]$$

$$-\frac{2}{\ell}\left[\underbrace{\left(2a-\frac{5a}{\ell}\cdot\frac{\ell}{5}\right)}_{2a-a=a}\frac{-\ell}{n\pi}\cos\frac{n\pi}{5} - \left(\frac{5a}{\ell}\right)\cdot\frac{\ell^2}{(n\pi)^2}\sin\frac{n\pi}{5}\right]$$

$$= \cancel{-\frac{2a}{n\pi}\cos\frac{n\pi}{5}} + \frac{10a}{(n\pi)^2}\sin\frac{n\pi}{5} - \frac{10a}{(n\pi)^2}\sin\frac{2n\pi}{5}$$

$$\cancel{+\frac{2a}{n\pi}\cos\frac{n\pi}{5}} + \frac{10a}{(n\pi)^2}\sin\frac{n\pi}{5} = \frac{10a}{(n\pi)^2}\left(2\sin\frac{n\pi}{5} - \sin\frac{2n\pi}{5}\right)$$

So: $$y(x,t) = \frac{10a}{\pi^2}\sum_{n=1}^{\infty}\underbrace{\frac{2\sin\frac{n\pi}{5} - \sin\frac{2n\pi}{5}}{n^2}}_{B_n}\sin\frac{n\pi}{\ell}x\cdot\cos\frac{n\pi}{\ell}vt$$

$$y(x,t) = \frac{10a}{\pi^2}\sum_{n=1}^{\infty}B_n\sin\frac{n\pi}{\ell}x\cdot\cos\frac{n\pi}{\ell}vt.$$

with $$B_n = \frac{2\sin\frac{n\pi}{5} - \sin\frac{2n\pi}{5}}{n^2}$$