DIVA: End-term test 2015

Write your name and student number, and write READABLE.

(1) 1. Solve the separable 1st order differential equation

$(x+xy)y'+y=0$ with the boundary condition $y(1) = 1$

(1) 2. We have the inhomogeneous 1st order differential equation

$dx+(x-e^{y})dy=0$

First, solve the homogeneous equation. *Hint: solve for x in terms of y.*

Then require $C \to C(y)$ and solve the inhomogeneous equation

3. The solution of an inhomogenous 2nd order LDE with constant coefficients $y''+py'+qy=r(x)$ (p,q constant) is $y(x)=y_c(x)+y_p(x)$

With $y_c(x)=c_1y_1(x)+c_2y_2(x)$ and $y_p(x)=u(x)y_1(x)+v(x)y_2(x)$

Under certain conditions it appears that:

$$u(x)=\int\frac{-y_2r(x)}{W(x)} \text{ en } v(x)=\int\frac{y_1r(x)}{W(x)} \text{ met } W(x)=y_1y_2'-y_1'y_2$$

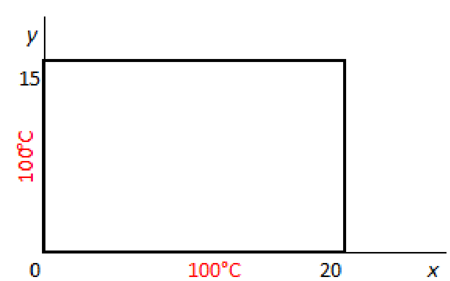
We can now solve any 2nd order LDE *(5 stappen plan)*

(1) a. $y''+16y=16\sin 4x$ **(1)** b. $2y''+y'=2x$

(2) 4. *Laplace:* $\nabla^2T=\frac{\partial^2T}{\partial x^2}+\frac{\partial^2T}{\partial y^2}=0$ with solutions $T=XY=\begin{Bmatrix}e^{ky}\\e^{-ky}\end{Bmatrix}\begin{Bmatrix}\sin kx\\\cos kx\end{Bmatrix}$

The steady state temperature distribution in a metal plate 10 cm square if one side (along x-axis) is held at 100°C and the other three sides at 0°C has a solution $T=\sum_{odd\ n}^{\infty}\frac{400}{n\pi\sinh n\pi}\sinh\frac{n\pi}{10}(10-y)\sin\frac{n\pi}{10}x$

Now consider a metal plate 15 x 20 cm square:



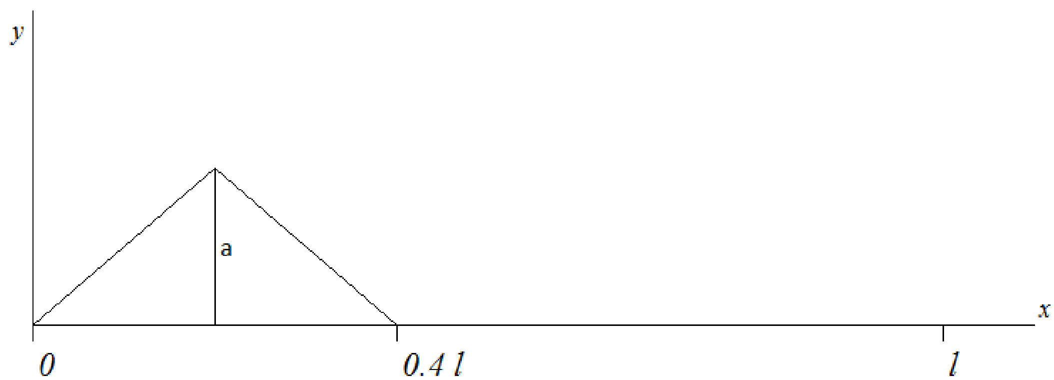
Find the steady state temperature distribution in this plate

(2) 5. *Diffusie equation:* $\nabla^2 u = \frac{1}{\alpha^2}\frac{\partial u}{\partial t}$ with solutions $u = FT = \begin{Bmatrix} e^{-k^2\alpha^2 t}\sin kx \\ e^{-k^2\alpha^2 t}\cos kx \end{Bmatrix}$

A bar of length l is initially at 0°C. From $t=0$ on, the $x=0$ end is held at T_1°C and the $x=l$ at T_2°C. Find the time-dependent temperature distribution.

(2) 6. *Wave equation:* $\nabla^2 y = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$ with solutions $y = XT = \begin{Bmatrix} \sin kx \\ \cos kx \end{Bmatrix}\begin{Bmatrix} \sin \omega t \\ \cos \omega t \end{Bmatrix}$

A string of length *l* has a zero initial velocity and a displacement as shown (plucked string problem)



Find the displacement as a function of *x* and *t*

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1 $\boxed{8.2.12}$. $(x+xy)y'+y=0 \rightarrow \frac{dy}{dx}=\frac{-y}{x(1+y)}$

$$\int \frac{1+y}{y}dy=\int -\frac{1}{x}dx \rightarrow \int \frac{1}{y}dy+\int dy=\int -\frac{1}{x}dx$$

$$\ln|y|+y=-\ln|x|+c \rightarrow \ln|y|+\ln|x|=-y+c$$

$$\ln|xy|=-y+c \rightarrow xy=e^{-y+c}=ke^{-y}$$

$$xye^{y}=k \qquad y=1 \text{ if } x=1 \rightarrow k=e$$

$$\rightarrow x=\frac{e}{ye^{y}}$$

2 $dx+(x-e^{y})dy=0$

$$\frac{dx}{dy}+x=e^{y}$$

1) $\frac{dx}{dy}+x=0 \rightarrow \int \frac{1}{x}dx=-\int dy$

$$\ln|x|=-y+c \rightarrow x_c=e^{-y+c_1}=ce^{-y}$$

2). Suppose $C=C(y) \rightarrow x=C(y)e^{-y}$

$$\frac{dx}{dy}+x=e^{y} \rightarrow c'(y)e^{-y}\cancel{-C(y)e^{-y}}\cancel{+C(y)e^{-y}}=e^{y}$$

$$\rightarrow c'(y)=e^{2y} \rightarrow C(y)=\frac{1}{2}e^{2y} \rightarrow x_p=\frac{1}{2}e^{2y}\cdot e^{-y}=\frac{1}{2}e^{y}$$

$$\boxed{x=x_c+x_p=ce^{-y}+\frac{1}{2}e^{y}}$$

3 a) $y'' + 16y = 16 \sin 4x$

see solution to $\boxed{8.6.17}$ homework
and replace $Re[\cdots]$ by $Im[\cdots]$

then: $y_p(x) = Im\left[-2ix\cos 4x + 2x\sin 4x + \frac{1}{4}\cos 4x + \frac{1}{4} i \sin 4x\right]$

$$= -2x\cos 4x + \frac{1}{4}\sin 4x.$$

hence: $y(x) = y_c + y_p$

$$= (d_1 + \tfrac{1}{4})\sin 4x + d_2\cos 4x + 2x\cos 4x$$

$$= C_1 \sin 4x + C_2 \cos 4x + 2x \cos 4x.$$

b) $\boxed{8.6.22}$ $\quad 2y'' + y' = 2x \quad$ First homogeneous:

$2m^2 + m = 0 \rightarrow 2m(m + \frac{1}{2}) = 0 \rightarrow m_1 = 0,\ m_2 = -\frac{1}{2}$

$\boxed{y_c} = c_1 e^0 + c_2 e^{-x/2} = \boxed{C_1 + C_2 e^{-x/2}} \rightarrow y_1 = 1, \quad y_2 = e^{-x/2}$

inhomogenous: $y_p = u(x) + v(x)e^{-x/2}$

$W(x) = 1 \cdot -\frac{1}{2}e^{-x/2} - 0 = -\frac{1}{2}e^{-x/2}$

$u(x) = \int \frac{e^{-x/2} \cdot \frac{2x}{2}}{-\frac{1}{2}e^{-x/2}} dx = \int 4x\, dx = 2x^2$

$v(x) = \int \frac{1 \cdot 2x}{-\frac{1}{2}e^{-x/2}} dx = -4\int x e^{x/2} = -4\left\{2xe^{x/2} - \int 2e^{x/2} dx\right\}$

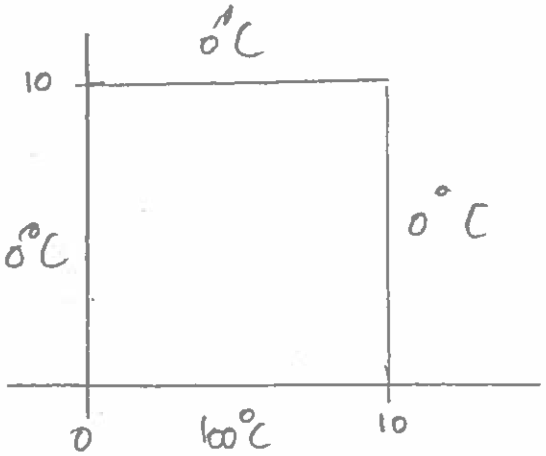
$= -8xe^{x/2} + 8 \cdot 2e^{x/2} = -8xe^{x/2} + 16e^{x/2}$

$y_p = 2x^2 + \left(-8xe^{x/2} + 16e^{x/2}\right)e^{-x/2} = 2x^2 - 8x + 16$

$y = y_c + y_p = C_1 + C_2 e^{-x/2} + 2x^2 - 8x + 16$

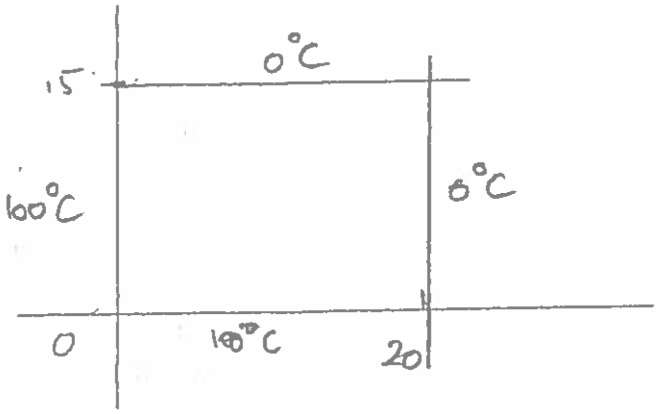
$\rightarrow y = A + Be^{-x/2} + x^2 - 4x. \quad \begin{cases} A = \frac{C_1}{2} + 16 \\ B = \frac{C_2}{2} \end{cases}$

(4)



Assignment $\boxed{3.2.10}$ gave the answer given in the exam. See also $\boxed{3.2.11}$ which was a home work assignment.

But now we have: (a variation of $\boxed{3.2.12}$)



So the solution for the x-axis 100°C becomes

$$T_1(x,y) = \sum_{\text{odd } n} \frac{400}{n\pi \sinh\left(\frac{3}{4}n\pi\right)} \cdot \sinh \frac{n\pi}{20}(15-y) \cdot \sin \frac{n\pi}{20}x$$

since $y = 15$ must give $T = 0°C$.

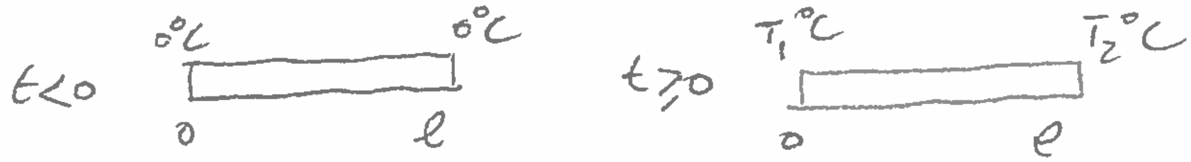
Similarly the solution for the y-axis 100°C is then:

$$T_2(x,y) = \sum_{\text{odd } n} \frac{400}{n\pi \sinh\left(\frac{4}{3}n\pi\right)} \cdot \sinh \frac{n\pi}{15}(20-x) \sin \frac{n\pi}{15}y$$

Hence, the final solution is $T(x,y) = T_1(x,y) + T_2(x,y)$

Note: in $T_1(x,y)$ the $\sinh\left(\frac{3}{4}n\pi\right)$ derives from the ratio $\frac{15}{20}$, in the $T_2(x,y)$ the $\sinh\left(\frac{4}{3}n\pi\right)$ derives from the ratio $\frac{20}{15}$

5. $\nabla^2 u = \frac{1}{\alpha^2}\frac{\partial u}{\partial t}$ met $u = FT = \begin{cases} e^{-k^2\alpha^2 t} \sin kx \\ e^{-k^2\alpha^2 t} \cos kx \end{cases}$



This is similar to Example 1 in Boas, pp. 629

$\left.\begin{array}{l} ax + b = T_1 \text{ op } x = 0 \rightarrow b = T_1 \\ ax + b = T_2 \text{ op } x = l \rightarrow al + b = T_2 \end{array}\right\} a = (T_2 - T_1)/l$

Hence; $u_0 = 0$, $u_f = \frac{T_2 - T_1}{l} x + T_1$

$\left.\begin{array}{l} \text{1. cos terms disappear} \\ \text{2. } \sin kl = 0 \rightarrow k = \frac{n\pi}{l} \end{array}\right\} u = e^{-\left(\frac{n\pi}{l}\right)^2 \alpha^2 t} \cdot \sin \frac{n\pi}{l} x$

3. At $t = 0$ is $u = u_0 \rightarrow \Sigma \ldots = u_0 - u_f = -u_f$.

Therefore: $b_n = \frac{2}{l}\int_0^l f(x) \sin kx \, dx$

$\rightarrow b_n = -\frac{2}{l}\int_0^l \left(\frac{T_2 - T_1}{l} x + T_1\right) \sin \frac{n\pi}{l} x \, dx$

$= -\frac{2(T_2 - T_1)}{l^2}\int_0^l x \sin \frac{n\pi}{l} x \, dx - \frac{2T_1}{l}\int_0^l \sin \frac{n\pi}{l} x \, dx$

Partial integration first part:

$\rightarrow b_n = -\frac{2(T_2 - T_1)}{l^2}\left[\frac{-x \, l}{n\pi} \cos \frac{n\pi}{l} x \Big|_0^l + \int_0^l \frac{l}{n\pi} \cos \frac{n\pi}{l} x \, dx \right.$

$= \frac{-2(T_2 - T_1)}{l^2}\left[-\frac{x l}{n\pi} \cos \frac{n\pi}{l} x + \cancel{\frac{l^2}{(n\pi)^2} \sin \frac{n\pi}{l} x} \right]_0^l$ (zero)

$= \frac{-2(T_2 - T_1)}{l^2}\left[-\frac{l^2}{n\pi} \cos n\pi \right] = \frac{2(T_2 - T_1)}{n\pi}(-1)^n$

5

ctd

Second part:

$$-\frac{2T_1}{\ell}\int_0^\ell \sin\frac{n\pi}{\ell}x\,dx = -\frac{2T_1}{\ell}\cdot\frac{-\ell}{n\pi}\left[\cos\frac{n\pi}{\ell}x\right]_0^\ell$$

$$= \frac{2T_1}{n\pi}\left[\cos n\pi - 1\right] = \frac{2T_1}{n\pi}\left[(-1)^n - 1\right]$$

$$= \begin{cases} 0 & n \text{ even} \\ \frac{-4T_1}{n\pi} & n \text{ odd} \end{cases}$$

Hence:

simplified:

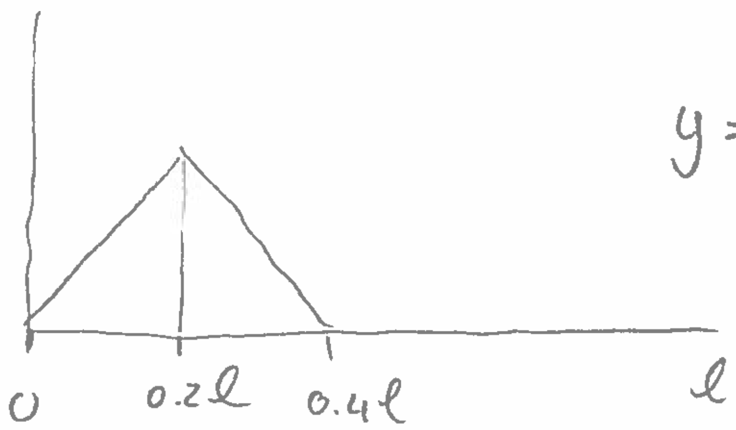
$$b_n = \frac{2(T_2-T_1)}{n\pi}(-1)^n - \frac{4T_1}{n\pi}\Big|_{n\,\text{odd}} = \frac{2}{n\pi}\left(T_2(-1)^n - T_1\right)$$

$$\rightarrow u = u_f + \frac{2(T_2-T_1)}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}e^{-k^2\alpha^2 t}\cdot\sin\frac{n\pi}{\ell}x$$

$$-\frac{4T_1}{\pi}\sum_{n\,\text{odd}}^{\infty}\frac{1}{n}e^{-k^2\alpha^2 t}\sin\frac{n\pi}{\ell}x$$

$$\rightarrow \boxed{u = T_1 + \frac{T_2-T_1}{\ell}x + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{T_2(-1)^n - T_1}{n}e^{-k^2\alpha^2 t}\cdot\sin\frac{n\pi}{\ell}x}$$

⑥



$$y = XT = \begin{Bmatrix} \sin kx \\ \cos kx \end{Bmatrix} \begin{Bmatrix} \sin \omega t \\ \cos \omega t \end{Bmatrix}$$

This is identical to $\boxed{13.4.2}$ with $0.2\,l \,/\, 0.4\,l$ instead of $\frac{1}{4}l \,/\, \frac{1}{2}l$, and $h = a$.

$$f(x) = \begin{cases} \frac{5a}{l}x & 0 < x < \frac{1}{5}l \\ 2a - \frac{5a}{l}x & \frac{1}{5}l < x < \frac{2}{5}l \\ 0 & \frac{2}{5}l < x < l \end{cases}$$

if $t=0$ then $\frac{\partial y}{\partial t} = 0$ so $\sin \omega t$ terms disappear

at $x=0 \rightarrow y=0$ so $\cos kx$ terms must go.

at $x = l \rightarrow y = 0 \rightarrow \sin kx = 0 \rightarrow k = \frac{n\pi}{l}$

$$y(x,t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{l} x \cdot \cos \frac{n\pi}{l} vt = f(x)$$

at $t=0$ $\cos \frac{n\pi}{l} vt = 1$ and hence:

$$b_n = \frac{2}{l} \int_0^l y(x,0) \sin \frac{n\pi}{l} x \, dx =$$

$$= \frac{2}{l} \int_0^{l/5} \frac{5a}{l} x \sin \frac{n\pi}{l} x \, dx + \frac{2}{l} \int_{l/5}^{2l/5} \left(2a - \frac{5a}{l}x\right) \sin \frac{n\pi}{l} x \, dx$$

6 ctd

$$b_n = \frac{2}{\ell}\cdot\frac{5a}{\ell}\left[\frac{-\ell}{n\pi}x\cos\frac{n\pi}{\ell}x + \frac{\ell^2}{(n\pi)^2}\sin\frac{n\pi}{\ell}x\right]_0^{\ell/5}$$

$$+\frac{2}{\ell}\left[\left(2a-\frac{5a}{\ell}x\right)\frac{-\ell}{n\pi}\cos\frac{n\pi}{\ell}x - \left(\frac{5a}{\ell}\right)\frac{\ell^2}{(n\pi)^2}\sin\frac{n\pi}{\ell}x\right]_{\ell/5}^{2\ell/5}$$

$$= \frac{10a}{\ell^2}\left[\frac{-\ell^2}{n\pi 5}\cos\frac{n\pi}{5} + \frac{\ell^2}{(n\pi)^2}\sin\frac{n\pi}{5}\right]$$

$$+\frac{2}{\ell}\left[\cancel{\underbrace{\left(2a-\frac{5a}{\ell}\frac{2\ell}{5}\right)}_{2a-2a=0}\frac{-\ell}{n\pi}\cos\frac{2n\pi}{5}} - 0 - \frac{5a}{\ell}\frac{\ell^2}{(n\pi)^2}\sin\frac{2n\pi}{5}\right]$$

$$-\frac{2}{\ell}\left[\underbrace{\left(2a-\frac{5a}{\ell}\cdot\frac{\ell}{5}\right)}_{2a-a=a}\frac{-\ell}{n\pi}\cos\frac{n\pi}{5} - \left(\frac{5a}{\ell}\right)\cdot\frac{\ell^2}{(n\pi)^2}\sin\frac{n\pi}{5}\right]$$

$$= \cancel{-\frac{2a}{n\pi}\cos\frac{n\pi}{5}} + \frac{10a}{(n\pi)^2}\sin\frac{n\pi}{5} - \frac{10a}{(n\pi)^2}\sin\frac{2n\pi}{5}$$

$$+\cancel{\frac{2a}{n\pi}\cos\frac{n\pi}{5}} + \frac{10a}{(n\pi)^2}\sin\frac{n\pi}{5} = \frac{10a}{(n\pi)^2}\left(2\sin\frac{n\pi}{5} - \sin\frac{2n\pi}{5}\right)$$

So: $$y(x,t) = \frac{10a}{\pi^2}\sum_{n=1}^{\infty}\underbrace{\frac{2\sin\frac{n\pi}{5} - \sin\frac{2n\pi}{5}}{n^2}}_{B_n}\sin\frac{n\pi}{\ell}x\cdot\cos\frac{n\pi}{\ell}vt$$

$$y(x,t) = \frac{10a}{\pi^2}\sum_{n=1}^{\infty} B_n \sin\frac{n\pi}{\ell}x\cdot\cos\frac{n\pi}{\ell}vt.$$

with $$B_n = \frac{2\sin\frac{n\pi}{5} - \sin\frac{2n\pi}{5}}{n^2}$$