DIVA: Mid-term test 1b - 2015

Write your name and write READABLE ! Assignments 1-3 are 1 point each, assignments 4-5 are 2 points, assignment 6 is 3 points

1. The charge q of a condensator varies with time. The corresponding current I is defined as $I = dq/dt$. Give the amplitude, period and frequency of q resp. I if:

   a) $q = f(t) = \mathrm{Re}\, 4e^{i24\pi t}$ b) $q = f(t) = \mathrm{Im}\, 3.5e^{i21\pi t}$

2. It appears that $\int_a^b \sin^2 kx\, dx = \int_a^b \cos^2 kx\, dx = \frac{1}{2}(b-a)$ <u>on the condition</u> that $k(b-a)$ is a multiple of π, or if both kb and ka are a multiple of $\frac{1}{2}\pi$. Knowing this, evaluate the following integrals:

   a) $\int_{\pi}^{4\pi/3} \cos^2 3x\, dx$ b) $\int_{-4/3}^{4/3} \sin^2(\frac{3\pi}{8}x)\, dx$

3. If you know that the Fourier series for $f(x) = \begin{cases} 0, & -\pi < x < 0 \\ 1, & 0 < x < \frac{1}{2}\pi \\ 0, & \frac{1}{2}\pi < x < \pi \end{cases}$ equals:

$$f(x) = \frac{1}{4} + \frac{1}{\pi}\left(\frac{\cos x}{1} - \frac{\cos 3x}{3} + \frac{\cos 5x}{5} + \ldots\right) + \frac{1}{\pi}\left(\frac{\sin x}{1} + \frac{2\sin 2x}{2} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \frac{2\sin 6x}{6} + \ldots\right)$$

   then give the Fourier series for $g(x) = \begin{cases} -\frac{3}{8}, & -\pi < x < 0 \\ \frac{3}{8}(\pi - 1), & 0 < x < \frac{1}{2}\pi \\ -\frac{3}{8}, & \frac{1}{2}\pi < x < \pi \end{cases}$

4. The complex Fourier series is given by $f(x) = \sum_{n=-\infty}^{n=+\infty} c_n e^{inx}$ with $c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx}\, dx$

   a) Determine the complex Fourier series for $f(x) = \begin{cases} 1, & -\pi < x < 0 \\ 0, & 0 < x < \pi \end{cases}$

   and b) write this as a sine-cosine Fourier series.

5. Instead of taking a period of 2π we can also develop a Fourier series over a period T, or over a wavelength $\lambda = 2l$. This gives for a_n :

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi}{l}x\,dx$$

a) Give an expression for b_n

b) If $f(x)$ is even or odd, what are the consequences for these coefficients ?

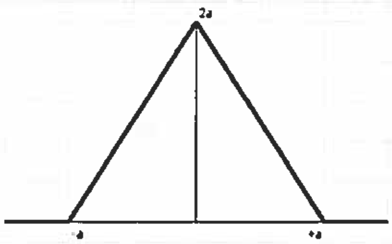
c) Determine if the following function is odd or even (make a sketch) and determine the Fourier series: $f(x) = \begin{cases} -1, & -l < x < 0 \\ 1, & 0 < x < l \end{cases}$

6. A Fourier integral is used to obtain a Fourier transform:

$$f(x) = \int_{-\infty}^{\infty} g(\alpha)e^{i\alpha x}d\alpha \text{ met } g(\alpha) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-i\alpha x}dx$$

a) Explain the difference with a Fourier series

b) Determine the function of the following graph:



c) Find the Fourier transform of the function

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1 Variation on $\boxed{7.2.12}$:

a) $q = f(t) = \mathrm{Re}\, 4e^{24i\pi t} = 4\cos(24\pi t)$

$\rightarrow A = 4 \quad T = \frac{1}{12} \quad f = 12$.

$I = \frac{dq}{dt} = -96\pi \sin(24\pi t) \rightarrow A_I = 96\pi$

b) $q = f(t) = \mathrm{Im}\, 3.5e^{-i21\pi t} = -3.5\sin(21\pi t)$

$\rightarrow A = 3.5 \quad T = \frac{1}{10.5} \quad f = 10.5$

$I = \frac{dq}{dt} = -73.5\pi\cos(21\pi t) \rightarrow A_I = 73.5\pi$

2 a) $k(b-a) = 3\left(\frac{4}{3}\pi - \frac{3}{3}\pi\right) = \pi$ hence:

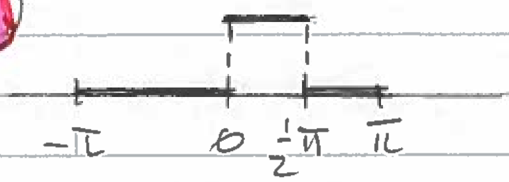
$\frac{1}{2}(b-a) = \frac{1}{2}\cdot\frac{1}{3}\pi = \frac{1}{6}\pi$.

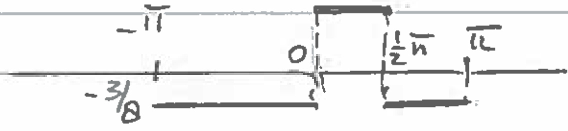
b) $k(b-a) = \frac{3}{8}\pi\left(\frac{4}{3} - \left(-\frac{4}{3}\right)\right) = \pi$ hence:

$\frac{1}{2}(b-a) = \frac{1}{2}\cdot\frac{8}{3} = \frac{8}{6} = \frac{4}{3}$

Variation on $\boxed{7.4.15}$

3



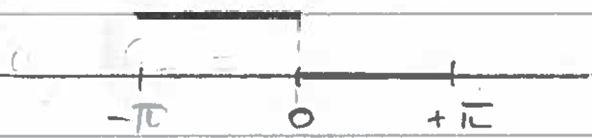


Hence: $g(x) = \frac{3}{8}\pi\, f(x) - \frac{3}{8}$

for $g(x)$ $\quad a_0 = \frac{3}{8}\pi \cdot \frac{1}{4} - \frac{3}{8} = \frac{3}{32}\pi - \frac{3}{8} = \frac{3}{32}(\pi - 4)$

$$\text{So } g(x) = a_0 + \frac{3}{8}\cancel{\frac{\pi}{\pi}}\left(\frac{\cos x}{1} - \frac{\cos 3x}{3} + \frac{\cos 5x}{5} - \dots\right) + \frac{3}{8}\cancel{\frac{\pi}{\pi}}\left(\frac{\sin x}{1} + \frac{2\sin 2x}{2} + \frac{\sin 3x}{3} + \dots\right)$$

4



$c_0 = \frac{1}{2}$ (graphical)

a) $$c_0 = \frac{1}{2\pi}\int_{-\pi}^{0} dx = \frac{\pi}{2\pi} = \frac{1}{2}$$ $\boxed{7.7.1}$

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{0} e^{-inx}\,dx = \frac{1}{2\pi}\left[\frac{e^{-inx}}{-in}\right]_{-\pi}^{0}$$

$$= \frac{i}{2\pi n}\left[1 - e^{in\pi}\right] = \begin{cases} \frac{i}{\pi n}, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$$

$$\rightarrow f(x) = \frac{1}{2} + \frac{i}{\pi}\sum \frac{e^{+inx}}{n}$$

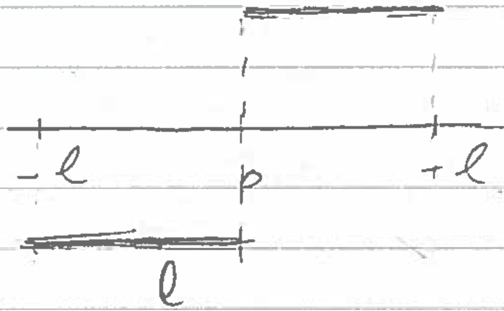
b) $$f(x) = \frac{1}{2} - \frac{2}{\pi}\left(\frac{e^{ix} - e^{-ix}}{2i} + \frac{e^{i3x} - e^{-i3x}}{3 \cdot 2i} + \dots\right)$$

$$= \frac{1}{2} - \frac{2}{\pi}\left(\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \dots\right)$$

COMPARE WITH PROBLEM $\boxed{7.5.1}$

5 $\boxed{7.9.6}$ $$f(x) = \begin{cases} -1 & -\ell < x < 0 \\ +1 & 0 < x < \ell \end{cases}$$

$a_0 = 0$



odd function

so $a_n = 0$, find b_n

$$b_n = \frac{2}{\ell}\int_0^{\ell} 1 \sin\left(\frac{n\pi x}{\ell}\right)dx = \frac{2}{\ell}\cdot\frac{\ell}{n\pi}\left[-\cos\frac{n\pi x}{\ell}\right]_0^{\ell}$$

$$= \frac{2}{n\pi}\left[-\cos n\pi + 1\right] = \begin{cases} 0, & n \text{ even} \\ \frac{4}{n\pi}, & \cancel{n \text{ odd}} \end{cases}$$

$$f(x) = \frac{4}{\pi}\left(\sin\left(\frac{\pi}{\ell}x\right) + \frac{1}{3}\sin\left(\frac{3\pi}{\ell}x\right) + \frac{1}{5}\sin\left(\frac{5\pi}{\ell}x\right)\dots\right)$$

$$= \frac{4}{\pi}\sum_{n\,\text{odd}}^{\infty}\frac{\sin\left(\frac{n\pi}{\ell}x\right)}{n}$$

a) $$b_n = \frac{1}{\ell}\int_{-\ell}^{+\ell} f(x)\sin\left(\frac{n\pi}{\ell}x\right)dx$$

b) $f(x)$ odd $\Rightarrow a_n = 0$, $b_n = \frac{2}{\ell}\int_0^{+\ell} f(x)\sin\left(\frac{n\pi}{\ell}x\right)dx$

$f(x)$ even $\rightarrow b_n = 0$, $a_n = \frac{2}{\ell}\int_0^{+\ell} f(x)\cos\left(\frac{n\pi}{\ell}x\right)dx$, 6 $\boxed{7, 12, 9}$

a) Fourier transformation is an integral, can handle all periods & frequencies, rather than a Fourier series which is the summation of (discrete) harmonics.

b)
$$f(x) = \begin{cases} 2(x+a) & -a < x < 0 \\ 2(a-x) & 0 < x < +a \\ 0 & \text{elsewhere} \end{cases}$$

c)
$$F(\alpha) = \frac{1}{\cancel{2}\pi}\left\{\int_{-a}^{0} \cancel{2}(x+a)e^{-i\alpha x}\,dx + \int_{0}^{+a} \cancel{2}(a-x)e^{-i\alpha x}\,dx\right\}$$

$$= \frac{1}{\pi}\cdot\frac{1}{-i\alpha}\left\{\left[(x+a)e^{-i\alpha x}\right]_{-a}^{0} - \int_{-a}^{0} e^{-i\alpha x}\,dx + \left[(a-x)e^{-i\alpha x}\right]_{0}^{+a} - \int_{0}^{+a} -e^{-i\alpha x}\,dx\right\}$$

$$= \frac{1}{\pi}\cdot\frac{1}{-i\alpha}\left[\cancel{a} + \frac{1-e^{+i\alpha a}}{i\alpha} - \cancel{a} - \frac{e^{-i\alpha a}-1}{i\alpha}\right]$$

$$= \frac{1}{\pi\alpha^2}\left[2 - e^{i\alpha a} - e^{-i\alpha a}\right] = \frac{2}{\pi\alpha^2}\left[1 - \cos\alpha a\right]$$

$$\rightarrow f(x) = \frac{2}{\pi}\int_{-\infty}^{+\infty} \frac{1-\cos\alpha a}{\alpha^2}\cdot e^{i\alpha x}\,d\alpha$$

But $f(x)$ is even function: $f(x) = \frac{4}{\pi}\int_{0}^{+\infty} \frac{1-\cos\alpha a}{\alpha^2}\cos\alpha x\,d\alpha.$