

DIVA: Mid-term test 1b - 2015

Write your name and write **READABLE** ! Assignments 1-3 are 1 point each, assignments 4-5 are 2 points, assignment 6 is 3 points

1. The charge q of a condenser varies with time. The corresponding current I is defined as $I = dq/dt$. Give the amplitude, period and frequency of q resp. I if:

a) $q = f(t) = \operatorname{Re} 4e^{i24\pi t}$ b) $q = f(t) = \operatorname{Im} 3.5e^{i21\pi t}$

2. It appears that $\int_a^b \sin^2 kx \, dx = \int_a^b \cos^2 kx \, dx = \frac{1}{2}(b-a)$ on the condition that $k(b-a)$ is a multiple of π , or if both kb and ka are a multiple of $\frac{1}{2}\pi$. Knowing this, evaluate the following integrals:

a) $\int_{\pi}^{4\pi} \cos^2 3x \, dx$ b) $\int_{-\pi/3}^{\pi/3} \sin^2\left(\frac{3\pi}{8}x\right) \, dx$

3. If you know that the Fourier series for $f(x) = \begin{cases} 0, & -\pi < x < 0 \\ 1, & 0 < x < \frac{1}{2}\pi \\ 0, & \frac{1}{2}\pi < x < \pi \end{cases}$ equals:

$$f(x) = \frac{1}{4} + \frac{1}{\pi} \left(\frac{\cos x}{1} - \frac{\cos 3x}{3} + \frac{\cos 5x}{5} + \dots \right) + \frac{1}{\pi} \left(\frac{\sin x}{1} + \frac{2 \sin 2x}{2} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \frac{2 \sin 6x}{6} + \dots \right)$$

then give the Fourier series for $g(x) = \begin{cases} -\frac{3}{8}, & -\pi < x < 0 \\ \frac{1}{8}(\pi - 1), & 0 < x < \frac{1}{2}\pi \\ -\frac{3}{8}, & \frac{1}{2}\pi < x < \pi \end{cases}$

4. The complex Fourier series is given by $f(x) = \sum_{n=-\infty}^{n=+\infty} c_n e^{inx}$ with $c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} \, dx$

a) Determine the complex Fourier series for $f(x) = \begin{cases} 1, & -\pi < x < 0 \\ 0, & 0 < x < \pi \end{cases}$

and b) write this as a sine-cosine Fourier series.

5. Instead of taking a period of 2π we can also develop a Fourier series over a period T , or over a wavelength $\lambda = 2l$. This gives for a_n :

$$a_n = \frac{1}{l} \int_{-l}^l f(x) \cos \frac{n\pi}{l} x dx \quad \text{a) Give an expression for } b_n$$

b) If $f(x)$ is even or odd, what are the consequences for these coefficients?

c) Determine if the following function is odd or even (make a sketch) and

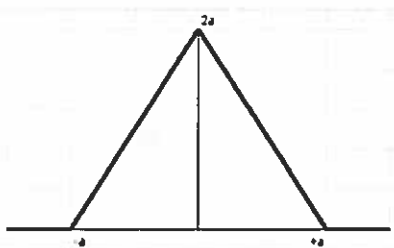
determine the Fourier series: $f(x) = \begin{cases} -1, & -l < x < 0 \\ 1, & 0 < x < l \end{cases}$

6. A Fourier integral is used to obtain a Fourier transform:

$$f(x) = \int_{-\infty}^{\infty} g(\alpha) e^{i\alpha x} d\alpha \quad \text{met} \quad g(\alpha) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-i\alpha x} dx$$

a) Explain the difference with a Fourier series

b) Determine the function of the following graph:



c) Find the Fourier transform of the function

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1 Variation on **7.2.12**:

$$a) q = f(t) = \operatorname{Re} 4e^{24i\pi t} = 4 \cos(24\pi t)$$

$$\rightarrow A = 4 \quad T = \frac{1}{12} \quad f = 12$$

$$I = \frac{dq}{dt} = -96\pi \sin(24\pi t) \rightarrow A_I = 96\pi$$

$$b) q = f(t) = \operatorname{Im} 3.5e^{-i21\pi t} = -3.5 \sin(21\pi t)$$

$$\rightarrow A = 3.5 \quad T = \frac{1}{10.5} \quad f = 10.5$$

$$I = \frac{dq}{dt} = -73.5\pi \cos(21\pi t) \rightarrow A_I = 73.5\pi$$

2 a) $k(b-a) = 3 \left(\frac{4}{3}\pi - \frac{3}{3}\pi \right) = \pi$ hence:

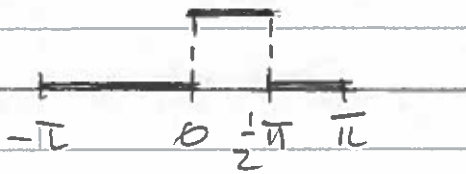
$$\frac{1}{2}(b-a) = \frac{1}{2} \cdot \frac{1}{3}\pi = \frac{1}{6}\pi$$

b) $k(b-a) = \frac{3}{8}\pi \left(\frac{4}{3} - \left(-\frac{4}{3}\right) \right) = \pi$ hence:

$$\frac{1}{2}(b-a) = \frac{1}{2} \cdot \frac{8}{3} = \frac{8}{6} = \frac{4}{3}$$

Variation on **7.4.15**

3



Hence: $g(x) = \frac{3}{8\pi} f(x) - \frac{3}{8}$

For $g(x)$ $a_0 = \frac{3}{8} \pi \cdot \frac{1}{4} - \frac{3}{8} = \frac{3}{32} \pi - \frac{3}{8} = \frac{3}{32} (\pi - 4)$

So $g(x) = a_0 + \frac{3}{8} \frac{\pi}{\pi} \left(\frac{\cos x}{1} - \frac{\cos 3x}{3} + \frac{\cos 5x}{5} - \dots \right)$
 $+ \frac{3}{8} \frac{\pi}{\pi} \left(\frac{\sin x}{1} + \frac{2 \sin 2x}{2} + \frac{\sin 3x}{3} + \dots \right)$

4



$c_0 = \frac{1}{2}$ (graphical)

a) $c_0 = \frac{1}{2\pi} \int_{-\pi}^0 dx = \frac{\pi}{2\pi} = \frac{1}{2}$

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$c_n = \frac{1}{2\pi} \int_{-\pi}^0 e^{-inx} dx = \frac{1}{2\pi} \left[\frac{e^{-inx}}{-in} \right]_{-\pi}^0$

$= \frac{i}{2\pi n} [1 - e^{in\pi}] = \begin{cases} \frac{i}{\pi n}, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$

$\rightarrow f(x) = \frac{1}{2} + \frac{i}{\pi} \sum \frac{e^{-inx}}{n}$

b) $f(x) = \frac{1}{2} - \frac{2}{\pi} \left(\frac{e^{ix} - e^{-ix}}{2i} + \frac{e^{i3x} - e^{-i3x}}{3 \cdot 2i} + \dots \right)$
 $= \frac{1}{2} - \frac{2}{\pi} \left(\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \dots \right)$

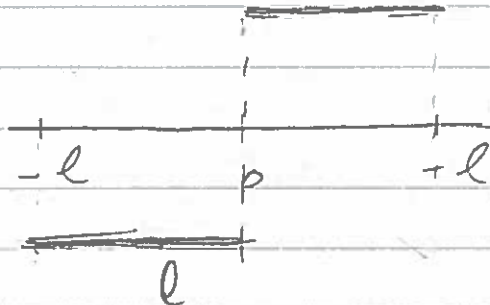
COMPARE WITH PROBLEM **7.5.1**

5

7.9.6

$$f(x) = \begin{cases} -1 & -l < x < 0 \\ +1 & 0 < x < l \end{cases}$$

$$a_0 = 0$$



odd function

so $a_n = 0$, find b_n

$$b_n = \frac{2}{l} \int_0^l 1 \sin\left(\frac{n\pi x}{l}\right) dx = \frac{2}{l} \cdot \frac{l}{n\pi} \left[-\cos \frac{n\pi x}{l} \right]_0^l$$

$$= \frac{2}{n\pi} \left[-\cos n\pi + 1 \right] = \begin{cases} 0, & n \text{ even} \\ \frac{4}{n\pi}, & n \text{ odd} \end{cases}$$

$$f(x) = \frac{4}{\pi} \left(\sin\left(\frac{\pi}{l}x\right) + \frac{1}{3} \sin\left(\frac{3\pi}{l}x\right) + \frac{1}{5} \sin\left(\frac{5\pi}{l}x\right) \right)$$

$$= \frac{4}{\pi} \sum_{n \text{ odd}}^{\infty} \frac{\sin\left(\frac{n\pi}{l}x\right)}{n}$$

$$a) \quad b_n = \frac{1}{l} \int_{-l}^{+l} f(x) \sin\left(\frac{n\pi}{l}x\right) dx$$

$$b) \quad f(x) \text{ odd} \Rightarrow a_n = 0, \quad b_n = \frac{2}{l} \int_0^{+l} f(x) \sin\left(\frac{n\pi}{l}x\right) dx$$

$$f(x) \text{ even} \Rightarrow b_n = 0, \quad a_n = \frac{2}{l} \int_0^{+l} f(x) \cos\left(\frac{n\pi}{l}x\right) dx$$

6

7.12.9

a) Fourier transformation is an integral, can handle all periods & frequencies, rather than a Fourier series which is the summation of (discrete) harmonics.

$$b) f(x) = \begin{cases} 2(x+a) & -a < x < 0 \\ 2(a-x) & 0 < x < +a \\ 0 & \text{elsewhere} \end{cases}$$

$$c) F(\alpha) = \frac{1}{2\pi} \left\{ \int_{-a}^0 2(x+a) e^{-i\alpha x} dx + \int_0^{+a} 2(a-x) e^{-i\alpha x} dx \right\}$$

$$= \frac{1}{\pi} \cdot \frac{1}{-i\alpha} \left\{ \left[(x+a) e^{-i\alpha x} \right]_{-a}^0 - \int_{-a}^0 e^{-i\alpha x} dx \right.$$

$$\left. + \left[(a-x) e^{-i\alpha x} \right]_0^{+a} - \int_0^{+a} e^{-i\alpha x} dx \right\}$$

$$= \frac{1}{\pi} \cdot \frac{1}{-i\alpha} \left[\cancel{a} + \frac{1 - e^{+i\alpha a}}{i\alpha} - \cancel{a} - \frac{e^{-i\alpha a} - 1}{i\alpha} \right]$$

$$= \frac{1}{\pi \alpha^2} \left[2 - e^{i\alpha a} - e^{-i\alpha a} \right] = \frac{2}{\pi \alpha^2} \left[1 - \cos \alpha a \right]$$

$$\Rightarrow f(x) = \frac{2}{\pi} \int_{-\infty}^{+\infty} \frac{1 - \cos \alpha a}{\alpha^2} \cdot e^{i\alpha x} d\alpha$$

But $f(x)$ is even function: $f(x) = \frac{4}{\pi} \int_0^{+\infty} \frac{1 - \cos \alpha a}{\alpha^2} \cos \alpha x d\alpha$