13/03/06

Midterm Examination "Geodynamics" March 6, 2013

problem: 1 The gravity potential of a spherically symmetric planet, $U(r)$, is described by Poisson's equation in the radial coordinate,

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dU}{dr}\right) = 4\pi G\rho \tag{1}$$

1. *Derive expressions for the gravity potential field U and the gravity force field $g = |\mathbf{g}|$ inside and outside the planet.*

   Hints: Solve Poisson's equation in spherical coordinates for the interior ($r \leq R$) and exterior domain $r \geq R$ separately. The separate solutions for the interior U_{int}, g_{int} and exterior U_{ext}, g_{ext} domain each contain two integration constants which can be determined by applying the following boundary conditions,

$$\lim_{r\to\infty} U_{ext}(r) = 0, \quad \lim_{r\to 0} g_{int}(r) < \infty \tag{2}$$

   Continuity of the gravity accelaration g at the surface $r = R$,

$$g_{int}(R) = g_{ext}(R) \tag{3}$$

   Continuity of the gravity potential U at the surface $r = R$,

$$U_{int}(R) = U_{ext}(R) \tag{4}$$

   Answers

$$g_{int} = \frac{4\pi}{3}G\rho_0 r\ , \quad U_{int} = \frac{2\pi}{3}G\rho_0 r^2 - \frac{3}{2}\frac{GM}{R} \tag{5}$$

   where $M = \frac{4\pi}{3}R^3\rho_0$ is the planet mass and G is the gravitational constant.

$$g_{ext} = \frac{GM}{r^2}\ , \quad U_{ext} = -\frac{GM}{r} \tag{6}$$

2. *Verify that the external gravity force field is identical to the field of a concentrated point mass at $r = 0$.*
3. *Derive an expression for the radial distribution of the pressure in the planetary interior and compute the central pressure for a case with $\rho_0 = 5.5 \cdot 10^3 \mathrm{kgm}^{-3}$ and $R = 6.371 \times 10^6 m$.*
   Solution: $P(r) = \frac{2\pi}{3}\rho_0^2 G\left(R^2 - r^2\right)$

problem: 2

The incompressibility K of a compressible medium is defined in terms of the density ρ and pressure P as,

$$\frac{1}{K} = \frac{1}{\rho}\frac{d\rho}{dP} \tag{7}$$

1. *Derive the Williamson-Adams equation (8) from the definition (1).*

$$\frac{d\rho}{dr} = -\frac{\rho^2 g}{K} \tag{8}$$

   where g is the gravity acceleration.

2. *Assume that both K and g are uniform and derive the following density-depth profile from (8),*

$$\rho(z) = \frac{\rho_s}{1 - \frac{\rho_s g z}{K}} \tag{9}$$

where $z = R - r$ is the depth coordinate, R the planetary radius and $\rho_s = \rho(0)$ the surface value of the density.

3. *Compute the depth where (9) becomes singular, that is where $1 - \rho_s g z / K$ becomes zero, for a model case with earth like parameters $K = 400$ GPa, $\rho_s = 3 \cdot 10^3$ kg/m^3, $g = 10$ m/s^2.*

4. *What would be the depth of the singularity for a large earth like planet of the same rock material as in the item 3, with mass $M_p = 8M_\oplus$ and radius $R_p = 1.5R_\oplus$? Assume the same surface density and incompressibility (not gravity) as in item 3.*

problem: 3

Consider conductive heat transport through a horizontal layer of thickness h and uniform thermal conductivity k, that is heated from below and cooled from above, with uniform bottom and top temperatures of $T_{top} = T_s$, $T_{bot} = T_s + \Delta T$.

Assuming a stationary situation with $\partial T / \partial t = 0$, heat transport in such a layer is described by the (Poisson type) heat diffusion equation $k\nabla^2 T + H_v = 0$, where H_v in the volumetric internal heating rate.

1. *Derive the vertical temperature profile through the layer for the special case without internal heating, $H_v = 0$.*

2. *Derive the corresponding profile for a layer with uniform internal heating, $H_v > 0$.*