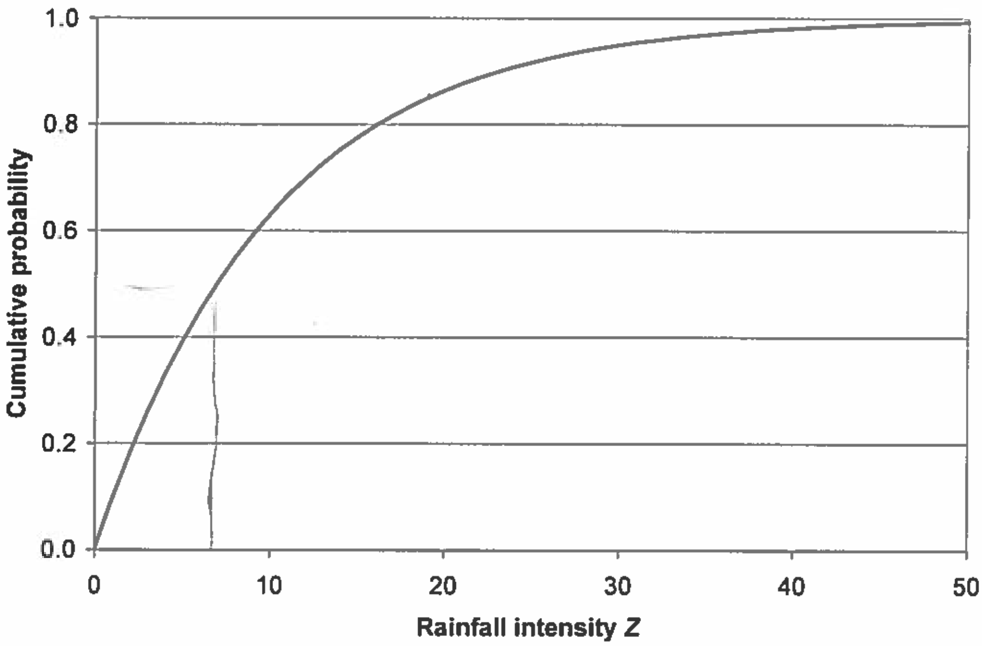
Exam GEO4-4420 Stochastic Hydrology

Monday April 18 13.30-16.30 Ruppert Rood

1. (3 points) Provide 5 reasons why it is advantageous to take uncertainty into account in hydrological modelling.

2. (6 points) Consider the following cumulative probability distribution function describing the rainfall intensity Z (mm/d) of a single rainfall event in a small catchment (see also Figure):

$$F_Z(z) = 1 - e^{-(z/10)}$$

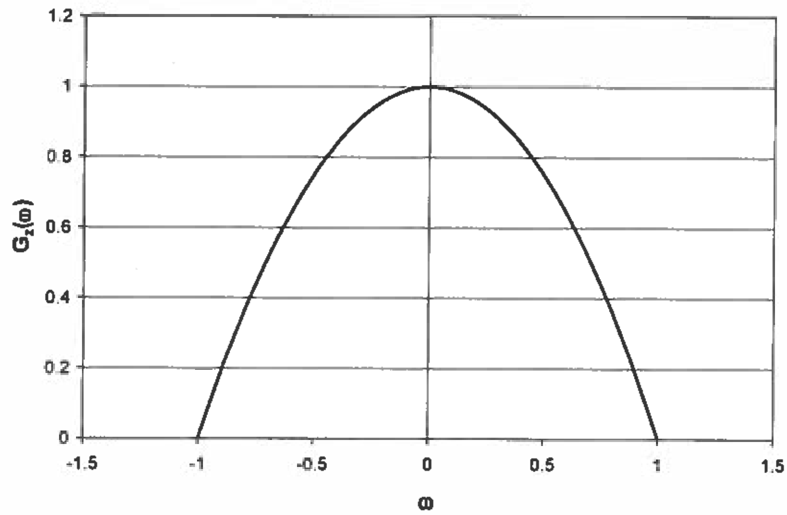


   a. What is the median rainfall intensity (calculate, not read from Figure)?
   b. Give an expression for the probability density function of rainfall intensity.
   c. Assuming that the infiltration capacity of the small catchment is homogeneous and equal to 20 mm/d, what is the probability that during a rainfall event runoff is generated?

3. (2 points) We consider a wide-sense stationary random function with the following spectral density function (see also Figure):

$$G_Z(\omega) = 1 - \omega^2$$

What is the variance of the random function?



4. (5 points) Consider a river where the mean and variance of yearly maximum discharge are respectively given by: μ_Y=1445 m^3/s and σ_Y^2 = 176646 m^6/s^2. These values have been obtained based on 80 years of observations.

a. Calculate the parameters a and b of the Gumbel distribution fitted to the data using the method of moments.

b. Estimate the value of flood that occurs on average every 1000 years and its 95% confidence interval.

c. What is the probability that this flood occurs at least once in the next 30 years?

5. (8 points) Consider the following isotropic covariance function of a wide-sense stationary function with mean ~~$\mu_Z = 5$~~:

$$C_Z(h) = \exp(-h/4)$$

$\mu_Z = 12$. Observations have been made at locations $z(x,y) = z(2,2) = 8$ and $z(x,y) = z(4,5) = 17$.

a. Use simple Kriging to predict the value of $Z(\mathbf{x})$ at location $(x,y) = (5,4)$ and estimate the prediction error variance.

b. If the mean was not known: which type of kriging would you use? Are there additional advantages to this type of kriging?

6. (8 points) Time series analysis/Kalman Filtering: Consider the following stochastic model describing monthly concentration of nitrogen c_t (in mg/l) in a lake with time steps of one month (index t is the month number):

$$C_t = 0.6C_{t-1} + 12q_t + W_t$$

With: q_t the monthly nitrogen input into the lake (10^3 kg /month) and

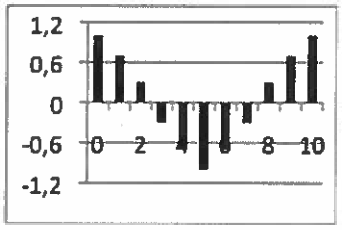
W_t a model error (system noise). The system noise is a white noise process.

The nitrogen input q_k is given in the following Table:

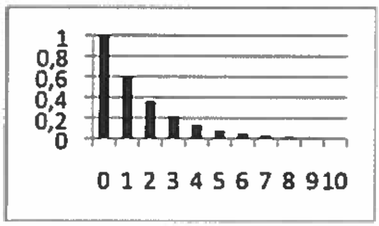
Time (month numbers)	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
N input (103 kg /month)	0	0	0	0	0	4	12	9	8	2	3	4	2	1	0

Before month number 6 there is no input of nitrogen (q_t=0, t<6). Therefore, for t<6 the model, describing the monthly concentration C_t, equals an ARIMA(1,0,0) model.

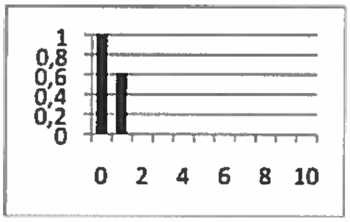
a. Below three autocorrelation functions (ACF) are given. Which one belongs to the ARIMA(1,0,0) model q_t?



ACF 1.



ACF 2.



ACF 3.

b. The concentration of nitrogen at month number 1 is given: c1 = 20 mg/l. The variance of the system noise is σ_W^2 = 36 mg ²/l². Calculate the forecast for the month numbers 2, 3 and 4 ($\hat{c}_{2|1}$, $\hat{c}_{3|1}$ and $\hat{c}_{4|1}$) using the ARIMA(1,0,0)-model, and the corresponding variance of the forecast error ($\sigma^2_{\hat{e}_{2|1}}$, $\sigma^2_{\hat{e}_{3|1}}$ and $\sigma^2_{\hat{e}_{4|1}}$). The forecast error variance for month number t is: $\sigma^2_{\hat{e}_{t|1}} = (0.6)^2 \cdot \sigma^2_{\hat{e}_{t-1|1}} + \sigma^2_W$

At month number 5 the estimate of the concentration is: $\hat{c}_{5|5}$=10 mg/l .The corresponding error variance is: $P_5 = \sigma^2_{\hat{e}_{5|5}}$ =9 mg2/l2. At t=8 and t=14 observations y_t are taken. We have y_8 = 165 mg/l and y_{14}= 79 mg/l. The variance of the observation error is the same for both times: $\sigma_V^2 = 16\text{mg}^2/\text{l}^2$.The measurement error v_t is a white noise process.

c. Apply the Kalman filter to obtain the optimal estimate $\hat{c}_t$ and the corresponding error variance $P_t = \sigma^2_{\hat{e}_{t|t}}$ for all time steps t = 6,..,15.

Equation sheet Stochastic Hydrology

Probability density $f_Z(z)$ *and cumulative probability* $F_Z(z)$

$$F_Z(z) = \Pr[Z \leq z] = \int_{-\infty}^{z} f_Z(z)dz$$

$$f_Z(z) = \frac{dF_Z(z)}{dz}$$

Note: For all integral involving infinity, the infinity sign is replaced by a maximum or minimum value if Z is bounded. For instance, if Z has a minimum value of 0 (e.g. hydraulic conductivity or concentration) the cumulative probability distribution function becomes:

$$F_Z(z) = \Pr[Z \leq z] = \int_{0}^{z} f_Z(z)dz$$

Some probability distributions:

<u>Discrete</u>

Binomial: Probability of n events in N trials, with probability p per trial:

$$\binom{N}{n} p^n (1-p)^{N-n}$$

$$n = 0,1,2,..,N$$

Geometric: Probability of number of trials n until the next occurrence of an event with probability p per trial:

$$(1-p)^{n-1} p$$

<u>Continuous</u>

Gaussian: $f_Z(z) = \frac{1}{\sigma\sqrt{\pi}} e^{-[\frac{1}{2}(z-\mu)^2/\sigma^2]}$

Exponential: $f_Z(z) = \lambda e^{-\lambda z}$

Mean μ_Z, *variance* σ_Z^2 and co-efficient of variation CV_Z *of a random variable Z*

$$\mu_Z = E[Z] = \int_{-\infty}^{\infty} z f_Z(z)dz$$

$$\sigma_Z^2 = E[(Z-\mu_Z)^2] = \int_{-\infty}^{\infty} (z-\mu_Z)^2 f_Z(z)dz = \int_{-\infty}^{\infty} z^2 f_Z(z)dz - \mu_Z^2$$

$$CV_Z = \frac{\sigma_Z}{\mu_Z}$$