Examination: Computational Geophysics
July 3, 2013, 13:15-16:15

Problem 1

Rayleigh-Benard (R-B) convection of an *isoviscous* fluid layer in the *Boussinesq-approximation* can be written interms of a streamfunction-vorticity formulation by the following coupled equations,

$$\nabla^2\omega = -Ra\partial_x T \tag{1}$$

$$\nabla^2\psi = \omega \tag{2}$$

$$\frac{\partial T}{\partial t} + \mathbf{u}\cdot\nabla T = \nabla^2 T \tag{3}$$

$\frac{\partial u_t}{\partial n} = 0$

We consider a 2-D rectangular domain $V = [0, L] \times [0, 1]$ with free-slip impermeable boundary conditions formulated as,

$\hookrightarrow u_n = \bar{u}\cdot n = 0$

$$\omega(\mathbf{x}) = 0, \psi(\mathbf{x}) = 0, \mathbf{x} \in \partial V \tag{4}$$

The model equations are discretized using a finite difference method on an equidistant grid of nodal points $\mathbf{x}_{ij} = (x_i, y_j)$.

1. Boundary conditions for the temperature field T in the R-B convection problem differ from the conditions (4) applied to ω, ψ problem. Specify apropriate boundary conditions for the temperature field T assuming symmetrical continuation of the finite domain in the horizontal direction.

2. Discuss the different number of degrees of freedom of the discretized solution fields $\Omega_{i,j} = \omega(\mathbf{x}_{ij})$, $\Psi_{i,j} = \psi(\mathbf{x}_{ij})$ on the one hand and $T_{i,j} = T(\mathbf{x}_{ij})$ on the other, as a result of the different boundary conditions for these fields.

3. In the following finite difference equations are considered for the coupled equations (1), (2), (3). First derive a system of finite difference equations for the vorticity equation (1) using central difference approximations for the Laplace operator and for the horizontal derivative $\partial_x T$.

   $$\mathbf{A\Omega} = \mathbf{F} \tag{5}$$

   Note: for consistency with the formulas in the following items the common prefactor h^{-2} in the discretized version of the Laplace operator should be included in the matrix coefficients of $\mathbf{A}$.

   Discuss the structure of the matrix of the resulting system of equations. (Symmetry, sparsity structure, bandwidth).

4. Derive a similar system of finite difference equations for the streamfunction equation (2), with the same matrix as in (5).

$$\mathbf{A}\Psi = \Omega \tag{6}$$

Explain why the matrices in (5) and (6) are identical and how the right hand side vector Ω is defined in terms of the vorticity field ω.

5. For the energy equation (3) we first consider a steady state model.

The velocity field in (3) is expressed in the streamfunction as,

$$\mathbf{u} = (\partial_z\psi, -\partial_x\psi) = -(\partial_y\psi, \partial_x\psi) \tag{7}$$

Discuss how the advection term in (3) can be discretized using central difference approximations for (7) and for the temperature gradient. How would you compute the velocity in points on the vertical boundaries? *Hint:* use an anti-symmetric continuation of the streamfunction in line with the homogeneous eassential boundary condition for the streamfunction.

The finite difference equations derived from the steady state version of (3) is written as,

$$\mathbf{BT} = \mathbf{R} \tag{8}$$

6. The discretized equations derived above can now be used to solve the steady state R-B convection problem. Explain why these three systems of equations are coupled in a non-linear way and discribe an iterative procedure for the solution of the coupled equations.

7. Explain the extension of the finite difference equations (8) into a system of ordinary differential equations by including a discretization of the time-dependent term in (3).

$$\mathbf{M}\frac{d}{dt}\mathbf{T} + \mathbf{BT} = \mathbf{R} \tag{9}$$

where $\mathbf{M} = h^2\mathbf{I}$, $\mathbf{I}$ is the identity matrix and h is the nodalpoint distance of the equidistant grid.

8. Derive the Euler forward and backward methods for the time integration of (9) and explain how the time dependent velocity field enters into the solution scheme.

9. Discuss relative merits of the two Euler schemes of item 8.