Examination: Computational Geophysics
July 4, 2004, 13:15-16:00, Room C118

problem 1.
Derive a finite difference formulation for the 2-D Laplace equation,

$$\nabla^2 u(x,y) = 0, \quad 0 \le x \le 2, \; 0 \le y \le 1, \tag{1}$$

Use a central difference approximation of the second order differential operator in (1) on an equidistant grid of $(N+2)\times(M+2)$ nodal points (including the boundary points), with grid point spacing h in both x and y and assume that essential boundary conditions are given in all the boundary points. Assume that the internal gridpoints of the finite difference mesh are numbered column wise in M columns of N points each.

1. *Derive an expression for a single matrix row and corresponding element of the righthand side vector of the finite difference equations.*

2. *Give a detailed description of the structure of the matrix of the finite difference equations. What is the bandwidth of the matrix expressed in the dimension of the grid.*

3. *Next extend the Laplace equation (1) into a differential equation for the time dependent temperature in a static (non-moving) medium including internal heating, assuming a uniform parameter for the thermal conductivity. Apply a semi-discretization to this time dependent heat equation, using the above method and grid for the Laplace operator, to derive a system of ordinary differential equations (ODE). Explain how you obtain the time-derivative term and the term describing the internal heating.*

4. *Describe how you would solve the system of ODE using the Crank-Nicolson scheme. Discus the implementation of boundary conditions in your solution, assuming similar time-constant essential boundary conditions as in the introduction.*

5. *Next consider a finite element discretization of (1) based on bi-linear quadrilateral elements on the same set of nodal points as in the finite difference grid. Show that the global finite element matrix build from the individual element matrices has a similar (not identical) structure as the finite difference matrix in the previous item. What is the bandwidth of the matrix and what is the number of non-zero coefficient in a matrix row?*

   Hint: First show that the element matrix is a 4×4 matrix by considering the expression for the element stiffness matrix in terms of general basis functions. Next consider the assembly procedure for building the global stiffness matrix from the element matrices and assume that the degrees of freedom are numbered in the same sequence as in the columnwise gridpoint numbering.

   Note: it is not necessary here to use explicit expressions for the coefficients of the element stiffness matrix.

problem 2. *The cooling of a static (non-moving) body can be described by the following time dependent heatconduction equation.*

$$\rho c_P \frac{\partial T}{\partial t} = \nabla \cdot k \nabla T + H \tag{2}$$

Consider the numerical solution of this equation for given initial values and boundary conditions for a three dimensional case.

1. *Apply a semi-discretization of (2) with a finite element method to derive the following system of ordinary differential equations,*

$$\mathbf{M}\frac{d}{dt}\mathbf{T} = \mathbf{S}\mathbf{T} + \mathbf{F} \tag{3}$$

   Assume general finite element basis functions $N_L(\mathbf{x})$ and a general non-uniform coefficient $\rho c_P(\mathbf{x})$. Explain the structure of the different objects in (3).

2. *Show how the mass matrix $\mathbf{M}$ can be approximated by a (lumped) diagonal matrix. Assume here a piecewise uniform approximation of the coefficient ρc_P on the elements.*

3. *Derive the Euler forward (explicit) and backward (implicit) schemes for the numerical integration of (3) and discuss their relative advantages and disadvantages in numerical applications.*

problem 3. *Consider the numerical modelling of a Raleigh-Benard (R-B) convection experiment with a viscous fluid layer heated from below and cooled from above.*
Give a general outline description (skipping technical detail) of the numerical solution of the coupled convection equations. Assume that the penalty function approach has been used to eliminate the pressure from the discretized flow equation. You may limit the problem to steady state R-B convection. How do you deal with the non-linear convective term in the heat transport equation?