Final examination: Computational Geophysics
June 26, 2015, 9:00-12:00

Problem 1

Consider the numerical solution of the steady state heat diffusion equation,

$$\nabla \cdot k\nabla T + H = 0 \tag{1}$$

on a domain V with boundary ∂V.

1. First consider the special case of the 1-D problem version of (1).
   Describe a finite difference method based on the finite volume concept that can be used to solve (1). In your answer include a treatment of different types of boundary conditions and of variable thermal conductivity k. How would you handle a model with discontinuities in k, for instance a horizontally layered configuration described by a 1-D model?
2. Next consider the general (1-D, 2-D or 3-D) problem version of (1).
   Apply the (Bubnov-Galerkin) finite element method to derive a system of linear algebraic equations for the temperature in the nodal points of a finite element mesh. Apply general basis functions $N_I(\mathbf{x})$ and use a general formulation independent of the dimension of the problem (1-D, 2-D or 3-D) in your derivation.
   Give expressions for the stiffnes matrix elements S_{IJ} and for the righthand side vector of the finite element equations interms of the general basisfunctions and the PDE coefficient functions k and H.
3. Derive expressions for the vector field of the heatflow density $\mathbf{J} = -k\nabla T$ in terms of the basis functions $N_I(\mathbf{x})$ and the solution vector $\mathbf{T}$ from the previous item. Discuss the computation of the surface heatfow through the domain boundary ∂V, from the heatflow vector $\mathbf{J}$.
4. Discuss the implementation of essential and natural boundary conditions, for the general problem of item 2.
5. Consider the special 1-D case of the above, on a domain $[0, z_{max}]$. Derive the foll-wing expressions for the stiffness matrix and righthand side vector of the finite element equations,

   $$S_{IJ} = \int_0^{z_{max}} k \frac{\partial N_I}{\partial z}\frac{\partial N_J}{\partial z} dz \tag{2}$$

   $$R_I = -\int_0^{z_{max}} H N_I dz - \left[\left(k\frac{\partial T}{\partial z}\right)_{z=0} \delta_{I1} - \left(k\frac{\partial T}{\partial z}\right)_{z=z_{max}} \delta_{IN}\right] \tag{3}$$

   where N is the number of nodal points.

6. Assume further that we use piece-wise linear basis functions $N_I(z)$ on the the 1-D domain containing N nodal points and $N-1$ elements and that we use a piece wise uniform (per element) thermal conductivity.
Derive the following expression for the stifness matrix for element number K,

$$\mathbf{S}^{(K)} = \frac{k_K}{h_K} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \tag{4}$$

7. Using (4) derive expressions for the global stiffness matrix, give an expression for the global matrix coefficients with row number I.

8. Apply the result of the previous item to a 1-D problem with mixed boundary conditions. Consider a case on a domain $[0,1]$, without internal heating $H=0$ and uniform conductivity $k=1$. Use an equidistant mesh with 4 nodalpoints and element size $h=1/3$. Apply the following boundary conditions: $T(z=0)=0$ and $k\partial T/\partial z(z=1)=1$. Derive the complete finite element equations for this problem. Solve the equations and compare your outcome with the exact solution of the partial differential equation, explain the identity of the two solutions.

9. Next consider the time dependent heat conduction problem extending the steady state equation (1) with the therm $\rho c_P \partial T/\partial t$. Show how the semi-discretisation of the time dependent heat equation with the (Bubnov-Galerkin) finite element method results in a system of ordinary differential equations. Use a general formulation valid for 1-D,2-D and 3-D problems.

10. Next consider the special 1-D case of the previous item and 1-D elements with basis functions as in item 6. Derive the following expression for the element mass matrices, assuming piecewise uniform heat capacity ρc_P per element,

$$\mathbf{M}^{(K)} = \frac{(\rho c_P)_K h_K}{6} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \tag{5}$$

Also derive expressions for the corresponding global mass matrix and in particular matrix row number I.

11. Discuss numerical integration techniques for the system of ODE's derived above. Derive an example of an explicit- and an implicit integration method and comment on their relative advantages and disadvantages.