3.13 EXAMPLE EXERCISE ON ELASTICITY PLUS WORKED ANSWER

Question

a) List the principal properties of isotropic and anisotropic elastic materials, and give examples of such materials.

b) Define the quantities "Young's Modulus" and "Poisson's Ratio" used to specify the elastic behaviour of isotropic materials.

c) Write down a set of equations giving the 3-D strain response of an isotropic elastic solid subjected to a state of stress defined by the principal stresses σ_1, σ_2, σ_3.

d) A block of isotropic quartzite (Young's Modulus E = 3 x 10^{10}Pa, Poisson's ratio $\nu = 0.3$) is subjected to a state of stress defined by the tensor

$$\sigma_{ij} = \begin{bmatrix} 90 & 0 & 0 \\ 0 & 60 & 0 \\ 0 & 0 & 30 \end{bmatrix} \text{MPa}$$

where compressive stress is taken as positive.

Calculate the resulting state of elastic strain, writing your answer in tensor notation.

Worked Answer

a) The principal properties of isotropic elastic materials are as follows:

- The elastic constants (such as Young's Modulus E or shear modulus G) are independent of direction.
- The principal stresses are parallel to the principal strains
- Under elevated hydrostatic pressure P, a sphere of elastically isotropic material compresses to a smaller sphere
- A bar of isotropic material can be stretched uniaxially without shearing, and can be bent without twisting

In contrast, elastically anisotropic materials possess the following principal properties:

- The elastic constants depend on orientation and are related to material symmetry.
- The principal stresses are generally not parallel to the principal strains.
- Under elevated hydrostatic pressure, a sphere becomes an ellipsoid.
- A bar of anisotropic material will generally shear when stretched and twist when bent.

Examples are given below.

- Elastically isotropic materials: polycrystalline solids with no crystallographic preferred orientation (e.g. statically recrystallized marble or metal), structureless rocks (e.g. massive quartzite or limestone).

- Elastically anisotropic materials: polycrystalline solids with a well-developed crystallographic preferred orientation (e.g. strongly deformed rocks or metals), foliated or layered rocks (e.g. slates or laminated sediments) anisotropically cracked materials, all single crystals.

b) Young's modulus (E) and Poisson's ratio (ν) are defined for uniaxial loading of an isotropic elastic material as follows:-

$E = \sigma_1/\varepsilon_1 = d\sigma_1/d\varepsilon_1 =$ slope of the linear stress (σ_1) versus axial strain (ε_1) curve obtained in uniaxial loading.

$\nu = -\varepsilon_1/\varepsilon_2$ where $\varepsilon_2 = \varepsilon_3$ is the lateral strain obtained in uniaxial loading.

c) The 3-D strain response of an elastically isotropic material is written

$$\varepsilon_1 = [\nu/E](\sigma_1/\nu - \sigma_2 - \sigma_3)$$

$$\varepsilon_2 = [\nu/E](\sigma_2/\nu - \sigma_1 - \sigma_3)$$

$$\varepsilon_3 = [\nu/E](\sigma_3/\nu - \sigma_1 - \sigma_2)$$

where the ε_i are the principal strains and the σ_i are the principal stresses.

d) For the block of quartzite, we have (given)

$E = 3 \times 10^{10}$Pa

$\nu = 0.3$

Furthermore, from the given values of σ_{ij} (and noting that the shear stress components are all zero) we can directly write the principal stresses as

$\sigma_1 = \sigma_{11} = 90$ MPa

$\sigma_2 = \sigma_{22} = 60$ MPa

$\sigma_3 = \sigma_{33} = 30$ MPa

Using the 3-D relations given under (c) hence leads to the result

$$\varepsilon_1 = \frac{0.3}{3\ .\ 10^{10}} \cdot \left(\frac{90}{0.3} - 60 - 30\right) .\ 10^6 = 2.1\ .\ 10^{-3}$$

$$\varepsilon_2 = \frac{0.3}{3\ .\ 10^{10}} \cdot \left(\frac{60}{0.3} - 90 - 30\right) .\ 10^6 = 0.8\ .\ 10^{-3}$$

$$\varepsilon_3 = \frac{0.3}{3\ .\ 10^{10}} \cdot \left(\frac{30}{0.3} - 90 - 60\right) .\ 10^6 = -0.5\ .\ 10^{-3}$$

which in tensor notation can be written

$$\varepsilon_{ij} = \begin{bmatrix} 2.1 & 0 & 0 \\ 0 & 0.8 & 0 \\ 0 & 0 & -0.5 \end{bmatrix} .\ 10^{-3}$$

4.11 MODEL ANSWER TO A PREVIOUS EXAM QUESTION

How to SCORE in your exam!

QUESTION

a) Explain what is meant by the term "failure criterion".

b) Write down the Coulomb criterion for shear failure of isotropic rock when pore fluid is present at pressure P_f. Identify all terms!

c) Write down an expression giving the failure plane orientation implied by the Coulomb criterion.

d) The earthquake of 1992 on the Peelrand fault near Roermond occurred at a depth of $h \approx 23$ km. Noting that the Peelrand fault is a normal fault bounding a graben structure, and assuming that the fault is characterized by a cohesive shear strength of 1 MPa and a coefficient of internal friction of 1/2, estimate the horizontal stress (σ_3) associated with the triggering of the earthquake. (These data are based on experimental data for long-inactive, healed faults).

HINT: Assume that σ_1 is subvertical and equal to the lithostatic overburden pressure ($p=\rho gh$) at h = 23 km depth, and that the pore fluid pressure at that depth is 0.8 p as frequently observed in boreholes. Take the density (ρ) of the overburden to be 2500 kg/m^3 and $g = 10$ ms^{-2}.

e) Comment on the depth at which the Roermond earthquake occurred.

ANSWER

a) A failure criterion is a mathematical relation specifying the states of stress (σ_{ij}) at which brittle failure of a material occurs.

b) The Coulomb criterion for shear failure of wet rock is written:

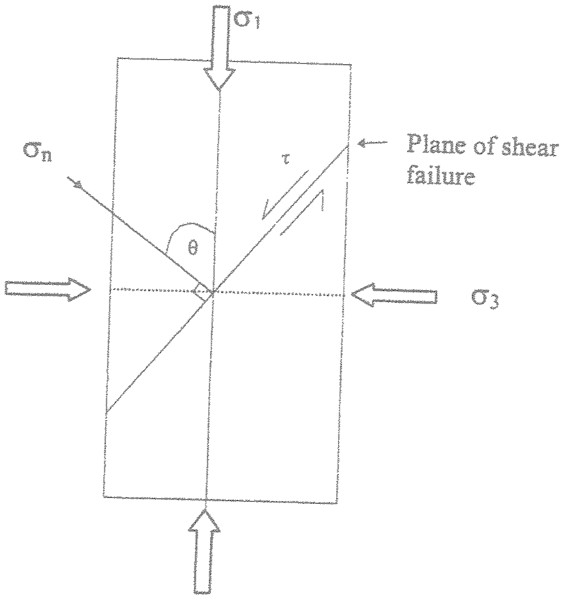
$$\tau = S_0 + \mu (\sigma_n - P_f)$$

where τ is the shear stress on the failure plane
S_0 is the cohesive shear strength of the rock
μ is the so-called coefficient of internal friction

σ_n is the normal stress acting across the failure plane
P_f is the pore fluid pressure

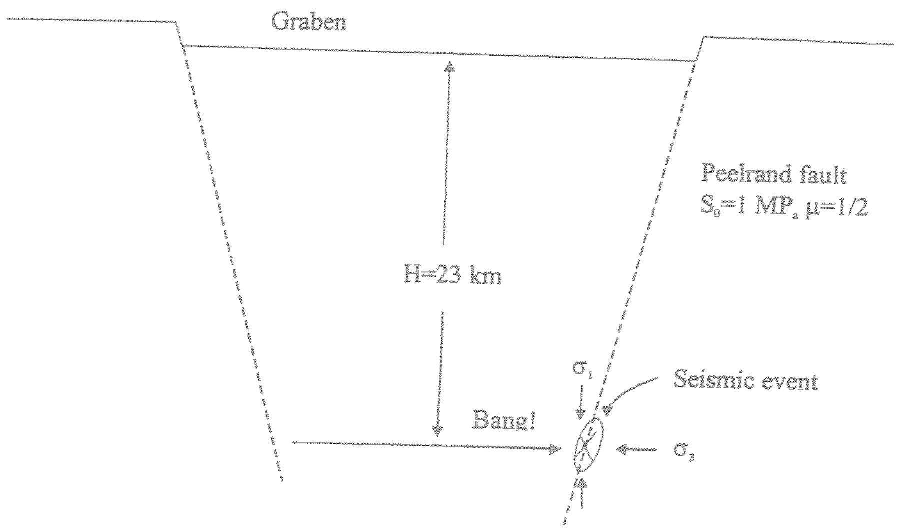
c) The failure plane orientation implied by the Coulomb criterion is given

$$\tan 2\theta = -\frac{1}{\mu}$$



where 2θ lies in the range $90° \leq 2\theta \leq 180°$
and θ is defined as shown above right.

d) <u>Problem:</u> Determine σ_3 associated with the triggering of seismic slip, in the following situation:-



From the given information, we have: $S_o = 1$ MPa and $\mu = 1/2$

Hence we can write, for the fault, the Coulomb criterion

$$\tau = 1 + (1/2)(\sigma_n - P_f) \quad \text{(MPa)} \qquad (1)$$

In addition, from the equations for 2-D stress we can write the following equations for the shear stress (τ) and normal stress (σ_n) on the fault plane

$$\tau = [1/2](\sigma_1 - \sigma_3)\sin 2\theta \qquad (2)$$

$$\sigma_n = [1/2](\sigma_1 + \sigma_3) + [1/2](\sigma_1 - \sigma_3)\cos 2\theta \qquad (3)$$

At failure, we must therefore have (from 1, 2, 3)

$$\frac{1}{2}(\sigma_1 - \sigma_3)\sin 2\theta = 1 + \frac{1}{2}\left\{\frac{\sigma_1 + \sigma_3}{2} + \frac{(\sigma_1 - \sigma_3)}{2}.\cos 2\theta - P_f\right\} \qquad (4)$$

Now, from the given information

σ_1 (vertical) = $\rho gh = (2500)(10)(23000) = 575$ MPA
$P_f = (0.8)\rho gh = 460$ MPa

and from the Coulomb criterion (eqn. 1)

$$\tan 2\theta = -\frac{1}{\mu} = -2 \quad (90° \leq 2\theta \leq 180°)$$

or $2\theta = 116.6°$ (students, be careful!! Note range of 2θ specified above)

Putting these values for σ_1, P_f and 2θ into (4) now yields

$$(575 - \sigma_3)(0.89) = 2 + \frac{575 + \sigma_3}{2} + \left(\frac{575 - \sigma_3}{2}\right)(-0.45) - 460$$

which on rearranging gives

$$\sigma_3 = 1623.25/3.23 = 503 \text{ MPa}$$

Hence the horizontal stress associated with triggering seismic slip on the fault is
$\sigma_3 = 503$ MPa ≈ 500 MPa

e) The Roermund earthquake occurred well below the depth of 10-15 km at which the brittle-ductile transition normally occurs. This could reflect either usually low values of σ_3 at depth (perhaps caused by an acceleration of crustal extension) or unusually high values of P_f. Both could cause brittle failure at depths where ductile behaviour is normally expected. Another possible explanation is an unusually low geothermal gradient, giving a deeper brittle-ductile transition. However, the geotherm is <u>not</u> unusually low in the Rhine-graben region.

4.12 EFFECT OF PORE FLUID PRESSURE ON ELASTIC BEHAVIOUR

4.12.1 Basic Theory

Before leaving the general topic of the behaviour of rock under upper crustal (elastic-brittle) conditions, let us consider the effects of pore fluid pressure (P_f) on elastic behaviour.

For the elastic response of isotropic rock in the absence of pore fluid pressure we have already had (taking compressive stresses and strains as positive)

$$2G\varepsilon_{ij} = \sigma_{ij} - \frac{1}{3}(1 - 2G/3K)\sigma_{kk}\delta_{ij} \qquad (1)$$

where σ_{ij} is the stress applied to the rock, ε_{ij} is the resulting strain, G is the shear modulus and K the bulk modulus of the ***bulk dry rock***. If the rock is porous and pore fluid is introduced at pressure P_f, the normal components of (effective) stress supported by the solid framework will be reduced by an amount P_f. Thus, the mean (effective) normal stress will change by an amount $\Delta\sigma = -P_f$. This will produce an isotropic elastic volumetric expansion strain given (from the definition of K) as

$$\varepsilon_v^- = \Delta\bar{\sigma} / K = -P_f / K = \varepsilon_{11}^- + \varepsilon_{22}^- + \varepsilon_{33}^-$$

$$\varepsilon_{ij}^- = -P_f\delta_{ij} / 3K \quad (\text{since } \varepsilon_{11}^- = \varepsilon_{22}^- = \varepsilon_{33}^-)$$

At the same time, P_f will cause an isotropic volumetric contraction of the grains (solid framework) of the rock given (from the definition of the bulk modulus of the grains, K_g)

$$\varepsilon_v^+ = P_f / K_g = \varepsilon_{11}^+ + \varepsilon_{22}^+ + \varepsilon_{33}^+$$

$$\varepsilon_{ij}^+ = P_f\delta_{ij} / 3K_g \,(\text{since } \varepsilon_{11}^+ = \varepsilon_{22}^+ = \varepsilon_{33}^+)$$