Midterm Examination "Geodynamics" March 7, 2012

The gravity potential of a spherically symmetric planet, $U(r)$, is described by Poisson's equation in the radial coordinate,

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dU}{dr}\right) = 4\pi G\rho \qquad (1)$$

problem: 1

1. *Derive expressions for the gravity potential field U and the gravity force field $g = |\mathbf{g}|$ inside and outside the planet.*

   Hints: Solve Poisson's equation in spherical coordinates for the interior ($r \leq R$) and exterior domain $r \geq R$ separately. The separate solutions for the interior U_{int}, g_{int} and exterior U_{ext}, g_{ext} domain each contain two integration constants which can be determined by applying the following boundary conditions,

   $$\lim_{r\to\infty} U_{ext}(r) = 0, \quad \lim_{r\to 0} g_{int}(r) < \infty \qquad (2)$$

   Continuity of the gravity accelaration g at the surface $r = R$,

   $$g_{int}(R) = g_{ext}(R) \qquad (3)$$

   Continuity of the gravity potential U at the surface $r = R$,

   $$U_{int}(R) = U_{ext}(R) \qquad (4)$$

   Answers

   $$g_{int} = \frac{4\pi}{3}G\rho_0 r\ , \quad U_{int} = \frac{2\pi}{3}G\rho_0 r^2 - \frac{3}{2}\frac{GM}{R} \qquad (5)$$

   where $M = \frac{4\pi}{3}R^3\rho_0$ is the planet mass and G is the gravitational constant.

   $$g_{ext} = \frac{GM}{r^2}\ , \quad U_{ext} = -\frac{GM}{r} \qquad (6)$$

2. *Verify that the external gravity force field is identical to the field of a concentrated point mass at $r = 0$.*

3. *Derive an expression for the radial distribution of the pressure in the planetary interior and compute the central pressure for a case with $\rho_0 = 5.5 \cdot 10^3 \mathrm{kgm}^{-3}$ and $R = 6.371 \times 10^6 m$.*

   Solution: $P(r) = \frac{2\pi}{3}\rho_0^2 G\left(R^2 - r^2\right)$

The lithostatic pressure can be expressed as the weight of a column of unit cross-sectional area extending from zero depth, at the Earth's surface, to the depth z of the evaluation point,

$$P(z) = \int_0^z \rho(z')g(z')dz' \qquad (7)$$

where ρ is the mass density and g is the magnitude of the gravitational acceleration.

problem: 2 *Derive the expression (7) (where the depth z is not to be confused with a cartesian coordinate) for the lithostatic pressure in a spherically symmetric planet from the Stokes equation for a static medium,*

$$\partial_j \sigma_{ij} + \rho g_i = 0 \tag{8}$$

Hint: Assume hydrostatic conditions where the stress tensor can be written as $\sigma_{ij} = -P\delta_{ij}$*, and derive from the Stokes equation (8) for the pressure gradient,* $\nabla P = \rho \mathbf{g}$.

problem: 3

1. *Derive the following equation for the temperature distribution of a W-A layer,*

$$\frac{dT}{dr} = -\frac{\alpha g}{c_P} T \tag{9}$$

where α *and* c_P *are the thermal expansion coefficient and the specific heat at constant pressure.*
Hint: Use the differential for the entropy,

$$dS = \left(\frac{\partial S}{\partial T}\right)_P dT + \left(\frac{\partial S}{\partial P}\right)_T dP \tag{10}$$

and the thermodynamic relations: $(\partial S/\partial T)_P = c_P/T$ *and* $(\partial S/\partial P)_T = -\alpha/\rho$.

2. *Derive the expression for the temperature profile for an adiabatic layer, sometimes referred to as the 'adiabat', by solving equation (9),*

$$T(r) = T(R) \exp\left(\int_r^R \frac{\alpha g}{c_P}\, dr'\right) \tag{11}$$

The temperature extrapolated to the surface, $T_P = T(R)$ *is known as the potential temperature of the W-A layer. The quantity* $H_T = (\alpha g/c_P)^{-1}$ *is known as the thermal scale height of the layer.*

3. *Derive an expression from (11) for the special case with a constant value of the scale height parameter.*

problem: 4 *Estimate the temperature near the bottom of the mantle by adiabatic extrapolation of the temperature* $T_{660} \sim 1900\mathrm{K}$ *of the phase transition near 660 km depth, to the depth of the core mantle boundary, using the general expression for the adiabat in a homogeneous layer.*

Hints: apply the result of **problem** *3 and assume uniform values of the 'scale height parameter'* $H_T = (\alpha g/c_P)^{-1}$*, with* $\alpha = 2 \cdot 10^{-5}\mathrm{K}^{-1}$*,* $g = 9.8\mathrm{ms}^{-2}$*,* $c_P = 1250\mathrm{Jkg}^{-1}\mathrm{K}^{-1}$*. Further: approximate the adiabat by a linear depth function, to obtain a uniform adiabatic temperature gradient.*