# Geodynamica 2019 - Midterm Exam

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## Exercise 1

Let us consider a spherically symmetric body of radius $R$.

1. (1 pt) as seen in class, use symmetry considerations with regards to the 3 axis to arrive at the the moment of inertia $I$:

$$I = \frac{8\pi}{3} \int_0^R \rho(r) r^4 dr \tag{1}$$

2. (1/2 pt) The density is given by

   - $\rho(r) = \rho_c$ for $r < R/3$,
   - $\rho(r) = \rho_{lm}$ for $R/3 \leq r < 5R/6$,
   - $\rho(r) = \rho_{um}$ for $r \geq 5R/6$.

   inside the planet and zero outside. Sketch $\rho(r)$ for $r \in [0 : \infty[$

3. (1 pt) Compute the total mass $M$ of the planet using this expression for the density.
4. (1 pt) Compute the moment of inertia $I$
5. (1/2 pt) Can $I$ be written $I = fMR^2$? if so, give $f$.
6. (1/2 pt) What are the dimensions of $\rho$, $I$ and $M$?

## Exercise 2

Let us consider a sphere of radius $R = 1$. The density of the material inside the sphere is given by:

$$\rho(r) = \rho_0(1 + r^n) \qquad n \geq 0$$

The gravitational potential satisfies the Poisson equation:

$$\Delta U = 4\pi \mathcal{G} \rho(r) \tag{2}$$

and we have the following relationship between the gravitational acceleration vector and the potential: $\boldsymbol{g} = -\boldsymbol{\nabla} U$.

1. (1/2 pt) sketch $\rho(r)$ inside the planet for various values of $n$. What does $\rho$ tend to for $n \rightarrow 0$ ? What does $\rho$ tend to for $n \rightarrow \infty$ ?
2. (1/2 pt) Write explicitely Eq.(2) for a point inside the sphere and a point outside the sphere.
3. (1 pt) Compute $g(r)$ and $U(r)$ for a point inside the sphere as a function of r (and n). Use

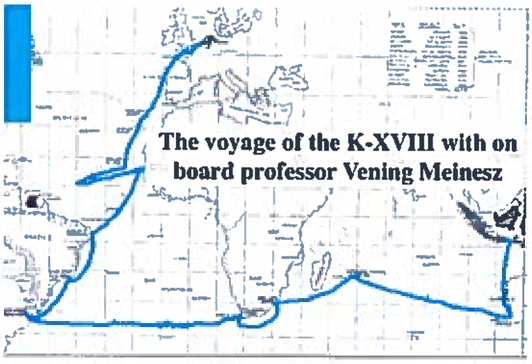
$$\lim_{r\rightarrow 0} g(r) \neq \infty$$

   to get rid an integration constant.

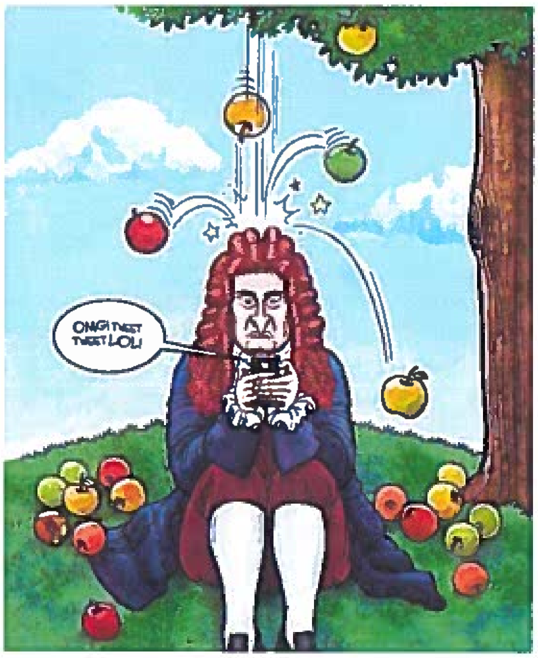
4. (1 pt) Same for a point outside the sphere.
5. (1 pt) Use the continuity of $g(r)$ and $U(r)$ at $r = R$ and $\lim_{r\rightarrow\infty} U(r) = 0$ to compute the last two remaining integration constants.
6. (1 pt) What do the expressions you have obtained for $g(r)$ and $U(r)$ both inside and outside the sphere tend to when $n \rightarrow \infty$ or $n \rightarrow 0$? Looking at the Appendix, conclude.

## Exercise 3

(1pt) This map shows the voyage of the submarine in which Prof. V. Meinesz performed his famous experiments. What kind of tectonic features did the submarine come across along the trip? How did these features and even the Earth' shape affect the measurements? Why was a submarine preferred to a (sail)boat?




## Exercise 4




(1 pt) Legend has it that Isaac Newton was sitting under an apple tree when he was hit on the head by a falling apple which prompted him to suddenly come up with his law of gravity. Although it did not happen this way, what kind of useful quantity could be measured with such an 'experiment'? What would then be the sources of uncertainty?

Appendix: Useful things to remember:
Elemental volume in spherical coordinates: $dV = r^2 \sin\theta dr\, d\theta\, d\phi$
Laplacian operator for spherically symmetric problems:

$$\Delta f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right)$$

Gravitational potential and gravity acceleration inside and outside a sphere of constant density $\rho_0$ and mass M.

$$g_{in}(r) = -\frac{4\pi}{3}\mathcal{G}\rho_0 r \qquad g_{out}(r) = -\frac{M\mathcal{G}}{r^2}$$

$$U_{in}(r) = \frac{2\pi}{3}\mathcal{G}\rho_0 r^2 - \frac{3}{2}\frac{M\mathcal{G}}{R} \qquad U_{out}(r) = -\frac{M\mathcal{G}}{r}$$