Midterm Examination "Geodynamics" March 12, 2014

Solve the following problems from the provided copied pages of the syllabus:

- Problem 18 (page 17+18), problem 21 (page 19)
- Problem 26 (page 22+23), problem 28 (page 28)

Extra problem

1. Discuss the geotherm in the core-mantle boundary region schematically illustrated in Fig.7 of the syllabus.
2. What are the implications of this particular geotherm for the dynamical state of the Earth's mantle.
3. Discuss the mineral phase transition from perovskite into the high-pressure form post-perovskite and explain the importance of seismic observation of this phase boundary in view of the previous items 1 and 2.

gravity acceleration and integrating the pressure gradient $dP/dr = -\rho g$. Assuming a zero pressure value at the surface this results in,

$$P(r) = \int_r^R \rho(r')g(r')dr' = 4\pi G \int_r^R \rho(r') \left\{ \frac{1}{r'^2} \int_0^{r'} \rho(r'')r''^2 dr'' \right\} dr' \quad (44)$$

The pressure in the Earth's interior reaches values over 350 GPa as shown in Fig. 1. For such high pressure values the effect of self-compression on the density is significant. In the following this effect is further explored.

The incompressibility K, or bulkmodulus [7] , is defined as,

$$\frac{1}{K} = \frac{1}{\rho}\frac{d\rho}{dP} \quad (45)$$

By substitution of $dP = -\rho g$ in (45) we derive an equation for the density profile,

$$\frac{1}{K} = \frac{-1}{\rho^2 g}\frac{d\rho}{dr} \Rightarrow \frac{d\rho}{dr} = -\frac{\rho^2 g}{K} \quad (46)$$

2.6.2 Parameterization of the bulkmodulus

The radial density distribution for a selfcompressing planet can be obtained from (46) once the bulkmodulus K is known. We will first consider simple cases where K is either a uniform constant or it is parameterized in terms of the density.

problem: 18 *Assume both K and g in (46) to be uniform in the mantle and derive the following density profile,*

$$\rho(z) = \frac{\rho_0}{1 - \frac{\rho_0 g z}{K}} \quad (47)$$

where $z = R - r$ is the depth coordinate and $\rho_0 = \rho(0)$ is the surface density value.

- *Compute the depth z_1 where this expression becomes singular, i.e. $\rho \to \infty$, suggesting infinite compression of the material. To do this assume Earth(mantle)-like values of the incompressibility, $K = 400$GPa (see Fig.3) and the surface density $\rho_0 = 3 \cdot 10^3$ kg/m^3.* [8]

[7] An isotropic linear elastic solid can be described by two independent elasticity parameters, for instance the Lamé parameters λ and μ. The bulkmodulus can be expressed in the Lamé parameters as, $K = \lambda + \frac{2}{3}\mu$ and the bulkmodulus K and the shearmodulus μ are the most commonly used parameters to specify the elastic parameters of Earth materials.

[8] Hint: First order ordinary differential equations like (46) are of so called separable form,

$$\frac{dy}{dx} = P(y)Q(x) \quad (48)$$

(see for instance, E.L. Ince, *Integration of ordinary differential equations*, Oliver and Boyd, 1956) in which case they can be integrated in the following way,

$$\frac{dy}{P(y)} = Q(x)dx \to \int \frac{dy}{P(y)} = \int Q(x)dx + C \to \quad (49)$$

In cases where the lefthand integral is a known function, say $f(y)$, the solution is obtained by the inverse function,

$$y(x) = f^{-1}\left(\int Q(x)dx + C\right) \quad (50)$$

Example: $dy/dx = -y^2 e^{-x}$, $x \geq 0$,

$$\int -\frac{dy}{y^2} = \int e^{-x}dx + C \to \frac{1}{y} = -e^{-x} + C \to y(x) = \frac{1}{C - e^{-x}} \quad (51)$$

The integration constant C can be expressed in an initial condition, $C = 1 + 1/y(0)$.

- *Now consider a simplified model of a large rocky exoplanet of Earth-like composition with $M = 8M_\oplus$ and $R = 1.5R_\oplus$. Assume uniform gravity (adapted for the given M, R) and uniform incompressibility K. Do you now find the singular depth z_1 within the depth range of the planet? Comment on the assumption of a uniform gravity field in view of the models presented in section 2.5.*

problem: 19 *The result of problem 18 gives the density depth distribution for the model with constant properties. The resulting expression (47) also contains the uniform gravity acceleration. A more fundamental relation between density and pressure can be derived for this model with constant material property as an equation of state (EOS) for the density.*

Derive the following logarithmic EOS for the density in terms of the static pressure, [9]

$$P = \ln\left(\left(\frac{\rho}{\rho_0}\right)^K\right) \tag{52}$$

The EOS (52) can be inverted to obtain an explicit expression for density as a function of pressure,

$$\rho(P) = \rho_0 \exp\left(\frac{P}{K}\right) \tag{53}$$

What happened to the singularity in (47) in this derivation?

The singular behavior in the above density model is a result of the assumed uniform g and K in (46). While g is reasonably constant with depth in the mantle, as illustrated in Fig. 1, K is not. The incompressibility increases with increasing depth/pressure and as a result the compression remains finite for earth-like conditions. The incompressibility can be expressed in the density and the seismic wave velocities, $v_p = \sqrt{(\lambda + 2\mu)/\rho}$, $v_s = \sqrt{\mu/\rho}$. With $K = \lambda + \frac{2}{3}\mu$ this becomes $K = \rho(v_p^2 - 4/3v_s^2)$. A radial profile $K(P(r))$ can therefore be derived, from the seismic velocities determined from inversion of traveltime tables of longitudinal and shearwave seismic arrivals.

The $K(P(r))$ profile derived from the PREM model of Dziewonski and Anderson (1981) appears to be roughly linear as shown in Fig.3.

A linear relation between bulkmodulus and pressure as suggested by Fig. 3, is also obtained using the following power law parameterization for the bulkmodulus in terms of the density $K(\rho)$.

$$K = C\rho^n \quad \Rightarrow \ln(K) = \ln(C) + n\ln(\rho) \quad \Rightarrow n = \frac{d\ln(K)}{d\ln(\rho)} = \frac{dK}{dP} = K_0' \tag{54}$$

where C is a constant. The constant pressure derivative in this model implies a linear pressure relation $K(P) = K_0 + K_0'P$. This appears to approximate the distribution of K in particular in the lower mantle as determined from seismological data in the PREM model. $K_0' \approx 4$ for the magnesium-iron sillicates $(Mg, Fe)SiO_3$ (perovskite) and dense oxides $(Mg, Fe)O$ (wüstite), representative for the earth's deep mantle. [10]

[9] Hint: evaluate the integral expression for pressure $P(z) = \int_0^z \rho g dz'$ by substitution of (47).

[10] An equation of state directly relating the density or specific volume to pressure can be derived from such an 'ansatz' of a linear pressure dependence $K = K_0 + K_0'P$ as shown in the following,

$$\frac{1}{\rho}\frac{d\rho}{dP} = \frac{1}{K} \rightarrow \frac{1}{V}\frac{dV}{dP} = -\frac{1}{K} \rightarrow dP = -(K_0 + K_0'P)\frac{1}{V}dV \tag{55}$$

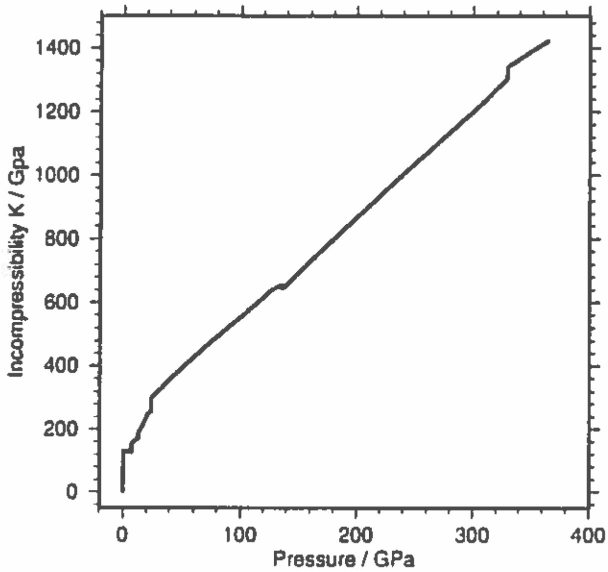

Figure 3: *Incompressibility profile derived from the PREM model.*

problem: 20 *Derive an explicit expression for the pressure dependent density from the Murnaghan equation of state (58).*
Answer:

$$\rho(P) = \rho_0 \left(\frac{K_0' P}{K_0} + 1 \right)^{1/K_0'} \tag{59}$$

problem: 21 *In the foregoing we have seen that a simple model with uniform incompressibility and gravity $K = K_0$ and $g = g_0$ leads to physically impossible solutions. In a refined version of this model, applied to the Earth's mantle, $g = g_0$ is maintained (compare Fig.1), and K is parameterized using the powerlaw relation (54).*
Derive the following density profile for the model corresponding to (54).

$$\rho(r) = \rho_0 \left(1 + (n-1) \frac{\rho_0 g_0 z}{K_0} \right)^{\frac{1}{n-1}} \tag{60}$$

where $z = R - r$ is the depth coordinate and the 0 subscript refers to zero pressure conditions. Note that the singularity for $\rho_0 g_0 z / K_0 = 1$ problem 18 is absent in this model.

$$\int_0^P \frac{dP'}{K_0 + K_0' P'} = -\int_{V_0}^{V} \frac{1}{V'} dV' = \int_V^{V_0} \frac{1}{V'} dV' = \ln\left(\frac{V_0}{V}\right) \tag{56}$$

Substitution in the integral over pressure of $K_0 + K_0'P' = x$, $dx = K_0' dP'$ gives,

$$\int_{x_0 = K_0}^{x_P = K_0 + K_0' P} \frac{1}{K_0'} \frac{dx}{x} = \frac{1}{K_0'} \ln\left(\frac{K_0 + K_0' P}{K_0}\right) = \ln\left(\frac{V_0}{V}\right) \tag{57}$$

$$1 + \frac{K_0' P}{K_0} = \left(\frac{V_0}{V}\right)^{K_0'} \rightarrow P = \frac{K_0}{K_0'} \left(\left(\frac{V_0}{V}\right)^{K_0'} - 1 \right) \tag{58}$$

This relation is known as the Murnaghan equation of state (EOS).

problem: 24 *Derive (67) by integration of the W-A equation (66).*

In (67) the gravity acceleration g depends on the density distribution $\rho(r)$ in the lefthand side. Therefore the density profile can not be simply obtained from a seismologically determined $\Phi(r)$ profile and a single evaluation of the integral in (67). The expression represents an integral equation that can be solved iteratively as specified in problem 25.

problem: 25 *Assume that a seismic parameter profile for the mantle $\Phi(r)$, obtained from seismic travel times, is available. Investigate how (67) can be used to compute a sequence of mantle density profiles $\rho^{(j)}(r), j = 1, 2, \ldots$ in an iterative procedure, by succesive substitution. How would you define a starting profile $\rho^{(1)}(r)$ for this iterative procedure?*
Hint: Substitute the density profile for iteration number j in the gravity acceleration in the righthand side of (67) for the computation of an updated profile $j+1$. This is an example of a general solution strategy for non-linear problems known as 'succesive substitution' or Picard iteration.

Williamson and Adams (1923) used the iterative scheme in problem 25 to test the hypothesis that the mass concentration towards the Earth's centre is completely explained by compression of a homogeneous self-gravitating sphere. They showed that integrating (67) from a surface value of $3.3 \cdot 10^3$ kg/m^3 results in unrealistically high density values for depths greater than the core-mantle boundary. This way they concluded that an inhomogeneous earth with a dense, compositionally distinct core, probably iron-nickle, was required by the observations. The necessary multiple integrals in the evaluation of (67) had to be computed by means of graphical approximation methods in 1923, several decades before the advent of electronic computers.

In a later analysis Bullen (1936) showed that the assumption of a homogeneous selfcompressing mantle described by the W-A equation, and a chemically distinct dense core, leads to unrealistically high values of the moment of inertia for the core $I_c = f M_c R_c^2$, with a prefactor value $f \sim 0.57$ greater than the value of a core with uniform density, 0.4. Since this would imply a density decrease towards the centre Bullen concluded that the applicability of the W-A model for the whole mantle can not be maintained and that instead a distinct mantle transition layer, labeled C-layer, must be included between the upper and lower mantle proper, related to transitions in mineral phase and/or composition (Bullen, 1975).

problem: 26

1. *Derive the following equation for the temperature distribution of a W-A layer (see Appendix A.3),*

$$\frac{dT}{dr} = -\frac{\alpha g}{c_P} T \tag{68}$$

where α and c_P are the thermal expansion coefficient and the specific heat at constant pressure.
Hint: Use the differential for the entropy,

$$dS = \left(\frac{\partial S}{\partial T}\right)_P dT + \left(\frac{\partial S}{\partial P}\right)_T dP \tag{69}$$

and the thermodynamic relations: $(\partial S/\partial T)_P = c_P/T$ and $(\partial S/\partial P)_T = -\alpha/\rho$.

2. *Derive the expression for the temperature profile for an adiabatic layer, sometimes referred to as the 'adiabat', by solving equation (68),*

$$T(r) = T(R) \exp\left(\int_r^R \frac{\alpha g}{c_P}\, dr'\right) \tag{70}$$

The temperature extrapolated to the surface, $T_P = T(R)$ is known as the potential temperature of the layer. The quantity $H_T = (\alpha g/c_P)^{-1}$ is known as the thermal scale height of the layer.

3. *Derive an expression from (70) for the special case with a constant value of the scale height parameter.*

The W-A equation for the density of an adiabatic layer can be generalized introducing the Bullen parameter η which is used as a measure of the departure of the actual density/temperature profile from an adiabat. This is done by writing,

$$\eta(r) = -\frac{\Phi}{\rho g}\frac{d\rho}{dr} \tag{71}$$

where $\eta(r)$ has been substituted for the constant value ($\equiv 1$) in the W-A equation.

2.7 Current density models

The concept of an adiabatic layer was essential when no independent determinations for the density distribution were available and the W-A equation was used to compute $\rho(r)$ for given values of the seismic parameter $\Phi(r)$ determined from seismological observations (Bullen, 1975).

During the 1970s a radial density distribution has been obtained for the Earth from inversion of seismological observations, incorporating spectral analysis of the Earth's eigenvibrations, under the constraints of the given values for M and I. This, together with seismic velocities determined from bodywave traveltimes and surfacewave dispersion, has resulted in the Preliminary Reference Earth Model (PREM), (Dziewonski and Anderson, 1981).

Since $\rho(r)$ can be determined from analysis of the earth's normal modes (radial eigenvibrations) the 'adiabaticity' of the mantle is no longer assumed.

The degree of 'adiabaticity' is used in numerical modelling experiments as a diagnostic for the dynamic state - where a high degree of adiabaticity indicates vigorous thermal convection and predominantly convective heat transport (van den Berg and Yuen, 1998, Matyska and Yuen, 2000, Bunge et al., 2001).

Usually the outcome of such experiments shows that the upper and lower mantle separately are approximately adiabatic - away from boundary layers were conductive transport dominates. In recent years models of the deep lower mantle have become popular were a compositionally distinct dense layer occupies the bottom 30% (roughly) of the lower mantle (Kellog et al., 1999, Albarede and van der Hilst, 2002).

Starting from these anchor points the temperature is then extrapolated from both sides to the core mantle boundary at 2900 km depth. For this temperature extrapolation assumptions have to be made about the dominant heat transport mechanism and in this case it is assumed that heat transport operates mainly through thermal convection. This will be further investigated in later sections dealing with heat transport in the Earth's mantle.

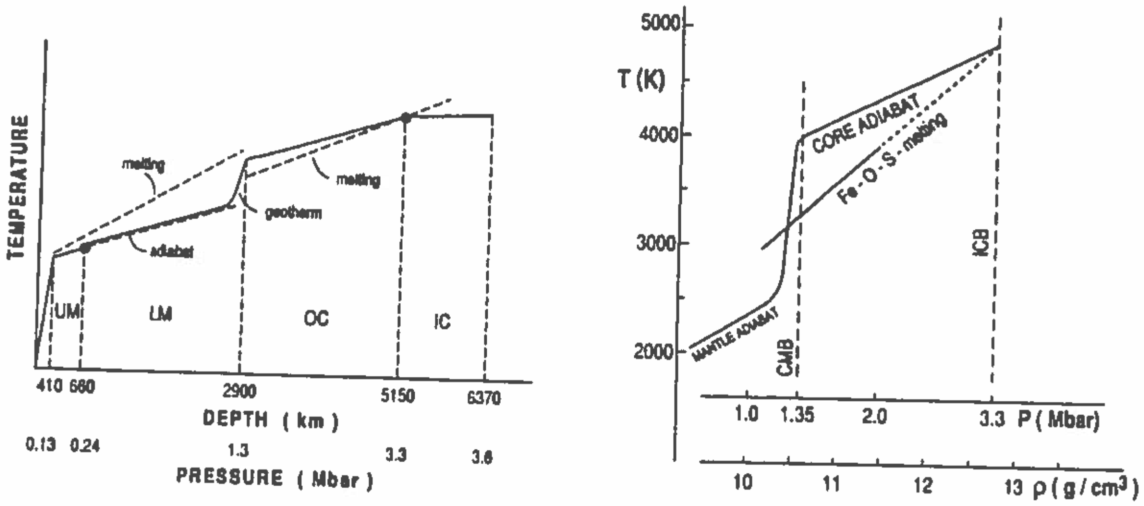

Figure 7: *Schematic radial temperature distribution in the mantle and core, constrained by major phase transitions. (UM-upper mantle, LM lower mantle, OC outer core, IC inner core). The temperature of the upper/lower mantle boundary is constrained by the γ-spinel to postspinel phase transition at 660 km depth. The temperature at the inner/outer core boundary at 5150 km depth (radius 1220 km) is constrained by the melting temperature of the hypothetical core 'Fe-O-S' alloy. The right hand frame shows a schematic core temperature distribution (geotherm) labeled 'CORE ADIABAT' in the liquid outer core versus pressure and the melting curve (liquidus) of the core 'Fe-O-S' alloy. (CMB core-mantle boundary, ICB inner core boundary). The ICB is determined by the intersection of the liquidus and the geotherm. During core cooling the ICB moves outward as the inner core grows by crystallisation.*

problem: 28 *Estimate the temperature near the bottom of the mantle by adiabatic extrapolation of the temperature T_{660} $\sim$ 1900K of the phase transition near 660 km depth, to the depth of the core mantle boundary, using the general expression for the adiabat in a homogeneous layer.*

Hints: apply the result of problem 26 and assume uniform values of the 'scale height parameter' $H_T = (\alpha g/c_P)^{-1}$, with $\alpha = 2 \cdot 10^{-5}\mathrm{K}^{-1}$, $g = 10\mathrm{ms}^{-2}$, $c_P = 1250\mathrm{Jkg}^{-1}\mathrm{K}^{-1}$. Further: approximate the adiabat by a linear depth function, in agreement with the schematic diagram of Fig. 7, to obtain a uniform adiabatic temperature gradient.