Midterm Examination "Geodynamics" March 4, 2015

Solve the following problems from the provided copied pages of the syllabus:

- Problem 15 (page 13+14) [1]
- Problem 29 (page 28). [2]

Extra problem

1. Discuss the geotherm in the core-mantle boundary region schematically illustrated in Fig.7 of the syllabus.
2. What are the implications of this particular geotherm for the dynamical state of the Earth's mantle.
3. Discuss the mineral phase transition from perovskite into the high-pressure form post-perovskite and explain the importance of seismic observation of this phase boundary in view of the previous items 1 and 2.
4. Discuss the phase transition of the olivine(ringwoodite) into perovskite plus magnesium-iron oxide(wuestite). Explain is its role in constraining the geotherm and also how this transition influences mantle convective circulation?

[1] For the reference to eqn. (15) the Poisson equation should be substituted, $\nabla^2 U = 4\pi G\rho$. For the special case of a spherically symmetric density distribution this can be written as: $\frac{1}{r^2}\frac{d}{dr}r^2\frac{dU}{dr} = 4\pi G\rho$.

[2] The adiabatic geotherm refered to in problem 29 can be formulated in terms of a depth coordinate as: $T_a(z) = T_a(0)\exp(z/H_T)$.

- A so called 'core catastrophe' occured when the iron/nickle core of the Earth differentiated from the silicate mantle in the first few million years after the formation of the Earth in the early solar system. This event has probably freed enough potential energy to melt the mantle completely, resulting in a global magma ocean.
- Crystallization of the solid inner core from the liquid outer core, as a result of core cooling, is accompanied by compositional differentiation. The liquid outer core contains a lighter fraction, possibly sulfur, which stays behind in the liquid during freezing of the inner core. The enriched residual liquid near the inner core boundary is less dense than the average liquid of the outer core and this results in a gravitationally unstable layering that induces 'chemically driven' convective flow in the outer core. The potential energy released in this chemical convection is probably an important energy source in powering the geodynamo that generates the Earth's present day magnetic field.

2.5 The gravity and pressure field for parameterized density models with self-gravitation

In the following problems a number of simple density distributions are investigated that will serve as a reference for models more constrained by geophysical observations to be introduced in later sections. The gravity field can be determined by solving the governing Poisson equation (15) using suitable boundary conditions. For the special case of spherically symmetric mass distributions simple 1-D integral expressions can be used to derive the corresponding radial pressure distribution.

problem: 15 *The internal and external gravity field for a simple model of a planet can by derived by solving the Poisson equation (15), and applying appropriate boundary conditions to the general solution. Consider a spherically symmetric planet of radius R and uniform density ρ_0.*

1. *Derive expressions for the gravity potential field U and the gravity force field $g = |\mathbf{g}|$ inside and outside the planet.*

   Hints: Solve Poisson's equation in spherical coordinates for the interior ($r \leq R$) and exterior domain $r \geq R$ separately. The separate solutions for the interior U_{int}, g_{int} and exterior U_{ext}, g_{ext} domain each contain two integration constants which can be determined by applying the following boundary conditions,

$$\lim_{r\to\infty} U_{ext}(r) = 0, \quad \lim_{r\to 0} g_{int}(r) < \infty \tag{26}$$

   Continuity of the gravity accelaration g at the surface $r = R$,

$$g_{int}(R) = g_{ext}(R) \tag{27}$$

   Continuity of the gravity potential U at the surface $r = R$,

$$U_{int}(R) = U_{ext}(R) \tag{28}$$

   Answers

$$g_{int} = \frac{4\pi}{3} G\rho_0 r \,, \quad U_{int} = \frac{2\pi}{3} G\rho_0 r^2 - \frac{3}{2}\frac{GM}{R} \tag{29}$$

   where $M = \frac{4\pi}{3} R^3 \rho_0$ is the planet mass and G is the gravitational constant.

$$g_{ext} = \frac{GM}{r^2} \,, \quad U_{ext} = -\frac{GM}{r} \tag{30}$$

2. *Verify that the external gravity force field is identical to the field of a concentrated point mass at $r = 0$. Derive a corresponding relation between the internal gravity force field and a (different) concentrated point mass, $m(r)$ at the center (see also (31)).*
3. *Derive an expression for the radial distribution of the pressure in the planetary interior and compute the central pressure for a case with $\rho_0 = 5.5 \cdot 10^3 \text{kgm}^{-3}$ and $R = 6.371 \times 10^6 m$.*
   Solution: $P(r) = \frac{2\pi}{3}\rho_0^2 G\left(R^2 - r^2\right)$

The gravity field of a spherically symmetric density distribution is identical to the field of an equivalent point-mass. This can be formulated as follows,

$$g(r) = \frac{Gm(r)}{r^2}, \; m(r) = \int_{V(r)} \rho dV = \int_0^r \rho(r')4\pi r'^2 dr' \tag{31}$$

Here $m(r)$ is the mass inside a sphere of radius r and $g(r)$ is the corresponding magnitude of the gravity acceleration. For the corresponding gravity potential this implies, with $\int_r^\infty \frac{dU}{dr'}dr' = U(\infty) - U(r) = -U(r)$,

$$U(r) = -\int_r^\infty \frac{dU}{dr'}dr' = \int_r^\infty g_r(r')dr' = \int_r^\infty -g(r')dr' = -\int_r^\infty \frac{Gm(r')}{r'^2}dr' \tag{32}$$

where the radial vector component g_r has been expressed in the vector length $g_r = \mathbf{g} \cdot \mathbf{e}_r = -g$.

To derive (31), the potential field at the radial coordinate r can be split in contributions originating from an internal- and external density distribution $U(r) = U_i(r) + U_e(r)$. With corresponding pairs, $U_i \leftrightarrow \rho_i$, and $U_e \leftrightarrow \rho_e$, where $\rho_e(r') = 0, \; r' \leq r$, and $\rho_e(r') = \rho(r'), \; r' > r$. This follows from the linearity of the governing Poisson equation.

The field generated by the internal mass distribution is obtained by integrating the corresponding Poisson equation in spherical coordinates,

$$\frac{1}{r'^2}\frac{d}{dr'}r'^2\frac{dU_i}{dr'} = 4\pi G\rho_i \tag{33}$$

$$\int_0^r \frac{d}{dr'}\left(r'^2\frac{dU_i}{dr'}\right)dr' = \int_0^r 4\pi G\rho_i r'^2 dr' \tag{34}$$

The radial component of the gravity acceleration becomes,

$$g_r(r) = -\frac{dU_i}{dr} = -\frac{1}{r^2}\int_0^r 4\pi G\rho_i r'^2 dr' = -\frac{Gm(r)}{r^2} \tag{35}$$

Furthermore the acceleration field g_e from the external mass distribution ρ_e for internal evaluation points $r' < r$ is zero. The corresponding gravity potential U_e is uniform, which follows from the relevant Poisson equation, in spherical coordinates for a spherically symmetric mass distribution,

$$\frac{1}{r'^2}\frac{d}{dr'}r'^2\frac{dU_e}{dr'} = 4\pi G\rho_e = 0 \;\rightarrow\; r'^2\frac{dU_e}{dr} = A \;\rightarrow\; g_e(r') = -\frac{dU_e}{dr'} = -\frac{A}{r'^2} \tag{36}$$

A non-singular field requires $A = 0, \; g_e(r') = 0, \; r' \leq 0$ and,

$$\frac{dU_e}{dr'} = 0 \;\rightarrow\; U_e(r') = B, \; r' \leq r \tag{37}$$

Starting from these anchor points the temperature is then extrapolated from both sides to the core mantle boundary at 2900 km depth. For this temperature extrapolation assumptions have to be made about the dominant heat transport mechanism and in this case it is assumed that heat transport operates mainly through thermal convection. This will be further investigated in later sections dealing with heat transport in the Earth's mantle.

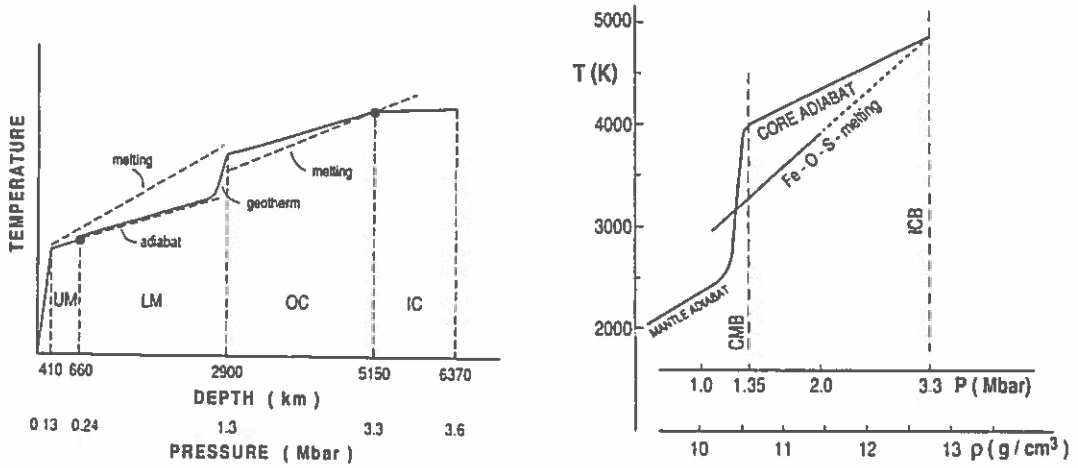

Figure 7: *Schematic radial temperature distribution in the mantle and core, constrained by major phase transitions (Boehler, 1996), (UM-upper mantle, LM lower mantle, OC outer core, IC inner core). The temperature of the upper/lower mantle boundary is construined by the γ-spinel to postspinel phase transition at 660 km depth. The temperature at the inner/outer core boundary at 5150 km depth (radius 1220 km) is constrained by the melting temperature of the hypothetical core 'Fe-O-S' alloy. The right hand frame shows a schematic core temperature distribution (geotherm) labeled 'CORE ADIABAT" in the liquid outer core versus pressure and the melting curve (liquidus) of the core 'Fe-O-S' alloy. (CMB core-mantle boundary, ICB inner core boundary). The ICB is determined by the intersection of the liquidus and the geotherm. During core cooling the ICB moves outward as the inner core grows by crystallisation.*

problem: 29 *Estimate the temperature near the bottom of the mantle by adiabatic extrapolation of the temperature* $T_{660} \sim 1900\mathrm{K}$ *of the phase transition near 660 km depth, to the depth of the core mantle boundary, using the general expression for the adiabat in a homogeneous layer.*

Hints: apply the result of problem 27 and assume uniform values of the 'scale height parameter' $H_T = (\alpha g/c_P)^{-1}$, *with* $\alpha = 2 \cdot 10^{-5}\mathrm{K}^{-1}$, $g = 10\mathrm{ms}^{-2}$, $c_P = 1250\mathrm{Jkg}^{-1}\mathrm{K}^{-1}$. *Further: approximate the adiabat by a linear depth function, in agreement with the schematic diagram of Fig. 7, to obtain a uniform adiabatic temperature gradient.*