(extra) Midterm Examination "Geodynamics" March 19, 2013

problem: 1 The gravity potential of a spherically symmetric planet, $U(r)$, is described by Poisson's equation in the radial coordinate,

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dU}{dr}\right) = 4\pi G\rho \tag{1}$$

1. *Derive expressions for the gravity potential field U and the gravity force field $g = |\mathbf{g}|$ inside and outside the planet.*

   Hints: Solve Poisson's equation in spherical coordinates for the interior ($r \leq R$) and exterior domain $r \geq R$ separately. The separate solutions for the interior U_{int}, g_{int} and exterior U_{ext}, g_{ext} domain each contain two integration constants which can be determined by applying the following boundary conditions,

   $$\lim_{r\to\infty} U_{ext}(r) = 0, \quad \lim_{r\to 0} g_{int}(r) < \infty \tag{2}$$

   Continuity of the gravity accelaration g at the surface $r = R$,

   $$g_{int}(R) = g_{ext}(R) \tag{3}$$

   Continuity of the gravity potential U at the surface $r = R$,

   $$U_{int}(R) = U_{ext}(R) \tag{4}$$

   Answers

   $$g_{int} = \frac{4\pi}{3}G\rho_0 r \ , \quad U_{int} = \frac{2\pi}{3}G\rho_0 r^2 - \frac{3}{2}\frac{GM}{R} \tag{5}$$

   where $M = \frac{4\pi}{3}R^3\rho_0$ is the planet mass and G is the gravitational constant.

   $$g_{ext} = \frac{GM}{r^2} \ , \quad U_{ext} = -\frac{GM}{r} \tag{6}$$

2. *Verify that the external gravity force field is identical to the field of a concentrated point mass at $r = 0$.*

3. *Derive an expression for the radial distribution of the pressure in the planetary interior and compute the central pressure for a case with $\rho_0 = 5.5 \cdot 10^3 \text{kgm}^{-3}$ and $R = 6.371 \times 10^6 m$.*

   Solution: $P(r) = \frac{2\pi}{3}\rho_0^2 G\left(R^2 - r^2\right)$

problem: 2

The relative inefficiency of thermal convection in mantle rock material can be illustrated clearly by the following thought experiment. Consider a purely conductive stationary model of a spherically symmetric Earth of radius R and uniform thermal conductivity k and with constant and uniform internal heating with a chondritic value $H = 5 \cdot 10^{-12}\mathrm{W/kg}$. For such a model the following steady state conductive heat equation applies,

$$k\nabla^2 T + \rho H = 0 \tag{7}$$

Rewrite the steady state equation (7) and derive the heat equation for an internally heated sphere in spherical coordinates,

$$\frac{k}{r^2}\frac{d}{dr}r^2\frac{d}{dr}T(r) + \rho H = 0 \tag{8}$$

Solve this equation and express the radial temperature profile in terms of the surface temperature $T(R) = T_R$ and the internal heating rate H as,

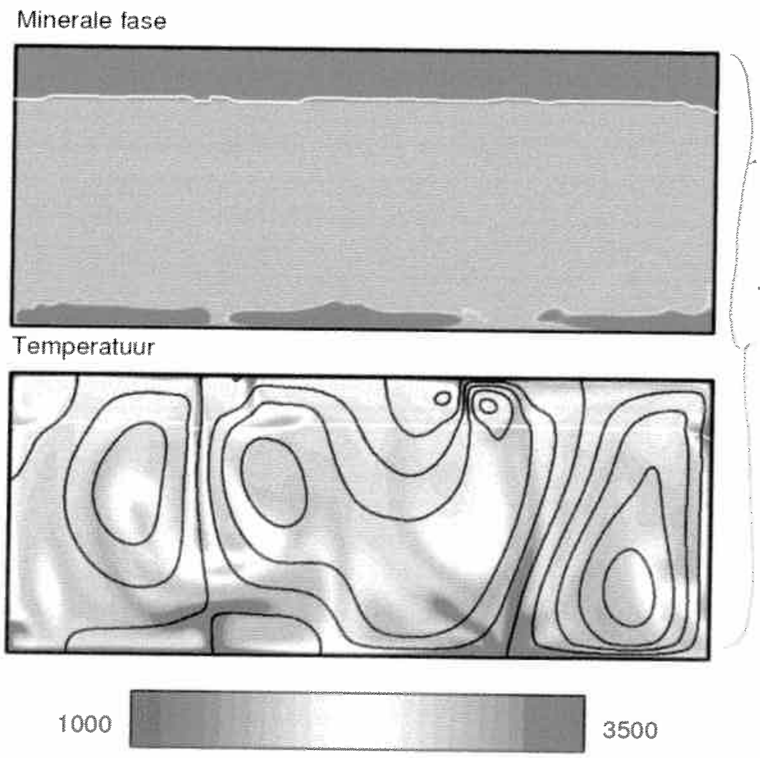
$$T(r) = T_R + \frac{\rho H}{6k}(R^2 - r^2) \tag{9}$$

Draw a schematic graph of the temperature distribution. Assuming earthlike values for the parameters, taking $k \sim 5\ \mathrm{WK^{-1}m^{-1}}$, compute the temperature value in the centre and halfway between the centre and the surface as an approximation of the core mantle boundary. What do you conclude from the outcome in view of the estimations of the internal temperature of the Earth based on experimentally determined phase transitions presented in in the lecture notes. What is the corresponding value of the surface heatflux and how does this compare to the estimated present day heatflux of the Earth?

problem: 3

Fig. 1 shows results of a model calculation for a simplified model of thermal convection in the Earth's mantle, that consists of an upper and a lower mantle separated by a mineral phase boundary.

Discuss the model results illustrated in this figure. What can you conclude about the nature of the phase transition involved, is it exothermic or endothermic? Motivate your answer by a consideration of the phase diagram of exothermic and endothermic phase transitions. What would be the effect of the phase boundary on mantle convection dynamics?



earths mantle

Figure 1: *Snapshot from a time dependent convection model showing mineral phase and temperature. The phases shown in the top frame are: blue olivine(spinel) green postspinel (perovskite plus periclase) and red postperovskite plus periclase. The white line marking the spinel to postspinel boundary defines the boundary between the upper and lower mantle near 660 km depth. The irregular structures near the bottom are regions occupied by the high pressure postperovskite phase.*