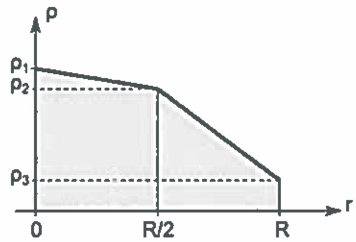
Mid-Term "Geodynamics", March 2016

Exercise 1

The moment of inertia I of a spherically symmetric body with outer radius R is given by

$$I = \frac{8\pi}{3}\int_0^R \rho(r) r^4 dr$$

The density distribution of this body is shown on the following figure:



1. Express the density profile shown on the figure by means of a function $\rho(r)$ composed of two continuous lines:

$$\rho(r) = \begin{cases} ar + b & r \leq R/2 \\ cr + d & r \geq R/2 \end{cases}$$

2. Compute the average density $< \rho >= \frac{1}{V}\int \rho dV$ of the planet as a function of $R, \rho_1, \rho_2, \rho_3$
3. Compute the momentum of inertia I for this density distribution
4. In the case where $\rho_1 = \rho_2 = \rho_3$, show that I can be written as $I = fMR^2$ where M is the mass of the body.

Exercise 2

The internal and external gravity field for a simple model of a planet can by derived by solving the Poisson equation:

$$\nabla^2 U = 4\pi \mathcal{G}\rho_0$$

and applying appropriate boundary conditions to the general solution. Consider a spherically symmetric planet of radius R and uniform density ρ_0.

1. Derive expressions for the gravity potential field U and the gravity force field $g = ||\mathbf{g}||$ inside and outside the planet.

   Hints: In the case of a spherically symetrical geometry, the Laplacian operator applied to a function $f(r)$ writes:

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right)$$

   Solve Poisson's equation in spherical coordinates for the interior ($r \leq R$) and exterior domain $r \geq R$ separately. The separate solutions for the interior U_{int}, g_{int} and exterior U_{ext}, g_{ext} domain each contain two integration constants which can be determined by applying the following boundary conditions,

$$\lim_{r\to\infty} U_{ext}(r) = 0, \quad \lim_{r\to 0} g_{int}(r) < \infty \tag{1}$$

   Continuity of the gravity accelaration g at the surface $r = R$, $g_{int}(R) = g_{ext}(R)$

   Continuity of the gravity potential U at the surface $r = R$, $U_{int}(R) = U_{ext}(R)$

   Answers

$$g_{int} = \frac{4\pi}{3}\mathcal{G}\rho_0 r\ , \quad U_{int} = \frac{2\pi}{3}\mathcal{G}\rho_0 r^2 - \frac{3}{2}\frac{\mathcal{G}M}{R}, \quad g_{ext} = \frac{\mathcal{G}M}{r^2}\ , \quad U_{ext} = -\frac{\mathcal{G}M}{r} \tag{2}$$

   where $M = \frac{4\pi}{3}R^3\rho_0$ is the planet mass and $\mathcal{G}$ is the gravitational constant.

2. Verify that the external gravity force field is identical to the field of a concentrated point mass at $r = 0$.
3. Derive an expression for the radial distribution of the pressure in the planetary interior and compute the central pressure for a case with $\rho_0 = 5.5 \cdot 10^3 \text{kgm}^{-3}$ and $R = 6.371 \times 10^6 m$.

   Solution: $P(r) = \frac{2\pi}{3}\rho_0^2\mathcal{G}\left(R^2 - r^2\right)$

   $G = 6.674 \cdot 10^{-11}$

Exercise 3

The Bulk modulus is defined as

$$\frac{1}{K} = \frac{1}{\rho}\frac{d\rho}{dP}$$

✱ Derive from this equation the following logarithmic equation of state in terms of the static pressure:

$$P = \ln((\rho/\rho_0)^K)$$

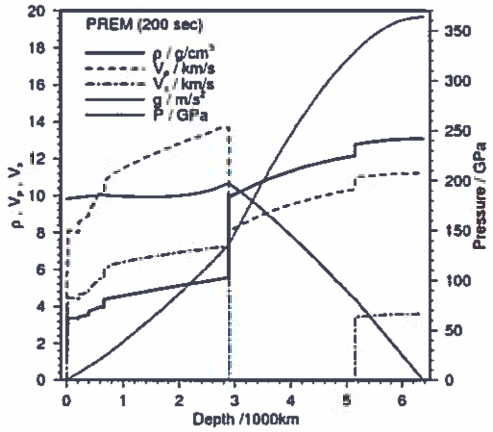
where you will specify the meaning of ρ_0. Show that we also have the following explicit expression for density as a function of pressure:

$$\rho(P) = \rho_0 \exp(P/K)$$

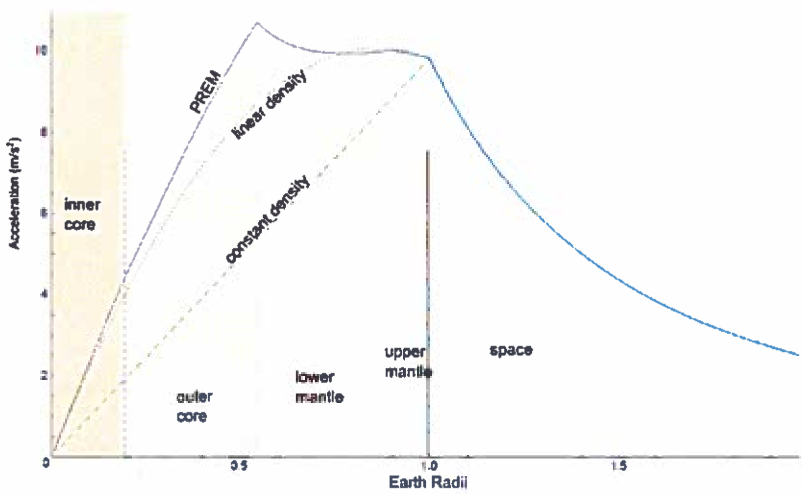
Sketch both these profiles.

Exercise 4

Discuss the curves pertaining to density, pressure and gravity as shown on the following figures. How would temperature effects alter these curves in space and/or time ?



Profiles as published in Dziewonski and Anderson in 1981.



Earth's Gravity according to the Preliminary Reference Earth Model (PREM). The acceleration has its maximum at approx 2890 km below the surface. The green curves show: (dashed) an idealized Earth with constant density (Earth's average density was used for this). (stippled) an approximated Earth, where the density decreases linearly from center to surface (the density at the center is the same as in the PREM, but the surface density is chosen in such a way that the mass of the resulting sphere equals the mass of the Earth). `https://en.wikipedia.org/wiki/Preliminary_reference_Earth_model`